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Research Article

Analytical free vibration solutions of Euler-Bernoulli beams resting on Winkler foundation using the Elzaki transform method

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ABSTRACT

The natural frequency analysis of a freely vibrating thin beam on Winkler foundation is a significant part of the analysis for design against resonance failures which occur when the natural frequency coincides with the excitation frequency due to loads. This article explores the natural frequency determinations of an Euler-Bernoulli beam on a Winkler foundation using Elzaki transform method (ETM). The main objective is to develop a new analytical solution procedure using the ETM and demonstrate its efficiency for beam foundation vibration problem. The ETM is adopted because of its effectiveness in solving both linear and partial differential equations problems, computational simplicity and versatility. For harmonic vibrations, the governing partial differential equation (GPDE) of motion is simplified to an ordinary differential equation (ODE) in terms of the space variable. The ETM further simplifies the ODE to an algebraic problem whose inverse yields the solution for the modal displacement in the physical domain. The application of boundary conditions gives the characteristic frequency equation as a transcendental equation. The roots give the eigenvalues from which the natural frequencies are found. The boundary conditions considered are the classical boundary conditions (BCs). The natural frequencies for all the classical BCs are found as exact solutions within the assumptions and theoretical framework of Euler-Bernoulli theory, Winkler foundation model, linear elastic behavior, small displacement elasticity theory and classical boundary conditions. They are identical with previous solutions using Generalized Integral Transform Methods (GITM). The present solutions are close to the previous solutions based on semi-analytical methods of Differential Transform Method (DTM). Analytical solutions for the natural frequencies are obtained for the classical BCs considered and for the foundation parameters considered. The effectiveness of the ETM in simplifying the natural frequency analysis of thin beam on Winkler foundation to an algebraic formulation is demonstrated.

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1. Introduction

1.1. Background

Beams are structural members with longitudinal dimensions that are much more than their cross-sectional dimensions. They are parts of building floors, bridges, foundation beams, machines, aircrafts and spacecrafts.

Beams that are placed on the ground are modeled using beam on elastic foundation theories. When such beams are subjected to dynamic forces, they can fail by resonance when the frequency of excitation forces becomes equal to the natural frequency of the beam on elastic foundation (BoEF) system. The calculation of the natural frequencies of vibration of BoEF systems is thus an important aspect of the analysis and design to avert resonance.

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Nomenclature

x	longitudinal coordinate
t	time
$r(x)$	foundation reaction
k	Winkler foundation stiffness
k_1	second parameter of the foundation
$w(x)$	transverse displacement (deflection)
u	Elzaki transform parameter
A	cross-sectional area
ρ	mass density
$q(x, t)$	applied load
T	Elzaki transform
I	moment of inertia
E	modulus of elasticity
ω_n	natural frequency
$4\beta^4$	foundation parameter defined in terms of k and EI
$K_0 = 4\beta^4 I^4$	foundation parameter
Ω_n^4	frequency parameter defined in terms of $\omega_n^2, \rho A, EI$
$W(u)$	Elzaki transform of $w(x)$
T^{-1}	Inverse Elzaki transform
l	length of beam
λ_n	dimensionless frequency parameter
α_n	parameter defined in terms of Ω_n^4 and $4\beta^4$
$M(x)$	bending moment
$Q(x)$	shear force
ETM	Elzaki transform method
DTM	differential transform method
GITM	generalized integral transform method
BCs	boundary conditions
ODE	ordinary differential equation
PDE	partial differential equation
BoEF	beam on elastic foundation
SVIM	Stodola-Vianello iteration method
ADM	Adomian decomposition method
GFEM	Galerkin finite element method
GDE	governing differential equation
FD	finite difference
FEMs	finite element method(s)
VIM	variational iteration method
DQM	differential quadrature method
DQEM	differential quadrature element method
HPM	homotopy perturbation method
EBBoWF	Euler-Bernoulli beam on Winkler foundation
C-free	clamped free
SS	simple supports
SC	simple support on left end and clamped on right
CC	clamped at both ends

The behaviours of beams have been described using several frameworks and models. The Euler-Bernoulli beam theory which is appropriate for slender beams is framed using the Euler-Bernoulli orthogonality hypothesis. This hypothesis is that the straight lines on the cross-section that are originally orthogonal to the middle surface of the beam before deformation remain straight and normal to the middle surface (plane) after deformation. This implies that shear stresses responsible for the change in shape of the cross-section is disregarded, limiting the scope of the resulting formulation to slender beams for which transverse shear deformations are negligible (Ike 2018a, 2018b, 2023).

Other beam theories have been formulated to consider shear deformations by Onah et al. (2020), Krishna Murty (1984), Ghugal (2007), and Sayyad and Ghugal (2016). Other thick beam models include sinusoidal shear deformable beams by Ike (2024a), hyperbolic shear deformable beams by Ike (2022), and first order shear deformable beams by Ike et al. (2019) and Ike (2019).

Kaniğ and Kutlu (2025) utilized a unified higher-order shear deformation theory (HSDT) that considered transverse stretching to analyze laminated composite plates. Kaniğ (2026a, 2026b) studied the stability of laminated composites by using a mixed finite element method based on a HSDT incorporating transverse stretching. The displacement field was defined using higher-order polynomial functions of thickness, leading to accurate expressions of both transverse shear and normal distributions without the need for shear modification factors. Moreover, Kaniğ and Kutlu (2026) developed a two-node C^0 -continuous mixed finite element formulation based on quasi-3D higher-order beam theory and the Hellinger-Reissner principle for the static bending analysis of laminated composite beams, incorporating transverse shear and normal deformation effects. Their work was however focused on beams and plates but with flexural analysis so they did not study on natural vibrations. On the other hand, recently, Kaniğ et al. (2026) developed a mixed HSDT finite element formulation based on the Hellinger-Reissner principle for the flexural analysis of laminated composites on Winkler-Pasternak elastic foundations. Also, Nezami (2026) proposed a compact higher-order shear and normal deformation theory with transverse stretching for the bending analysis of laminated composite plates, achieving accurate displacement and stress predictions with a fixed number of displacement variables independent of the number of layers.

Elastic foundation models have been proposed and developed by Winkler, Vlasov, Hetenyi, Kerr, Pasternak and Filonenko-Borodich. The Winkler foundation model is the simplest representation of the soil interaction with the beam using one-parameter, called the stiffness of the foundation. The model assumes that the foundation reaction $r(x)$ at any point on the beam is directly proportional to the beam deflection $w(x)$ at the point, and the proportionality constant (k) is the soil's stiffness (Ike 2018a, 2018b). Hence,

$$r(x) = kw(x) \quad (1)$$

This is equivalent to a representation of the foundation using a model of closely spaced vertical elastic springs which are unconnected to one another such that interactions do not exist between the springs. The Winkler foundation model is represented in Fig. 1.

The Winkler foundation model however cannot sustain shear stress and causes a discontinuity of deflec-

tions outside the loaded beam. The value of the Winkler stiffness parameter is determined by the geometry, foundation dimensions and soil stratification. The model gave no systematic estimation of the Winkler parameter apart from possibility of empirical determination via loading tests (Balabušić et al. 2019; Yelce et al. 2023; Arifin et al. 2020; Sadeghian and Ekhteraei Toussi 2011).

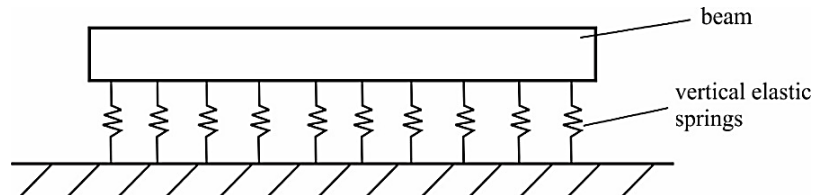


Fig. 1. Winkler foundation model represented as a bed of closely spaced springs.

These obvious limitations of the one-parameter model led to the development of two-parameter models shown in Fig. 2, by Vlasov, Hetenyi, Filonenko-Borodich and Pasternak (Doeva et al. 2022; Akhazhanov et al. 2023a, 2023b). These two-parameter models were developed primarily to resolve the displacement discontinuity issues of the Winkler model by incorporating a second interacting/coupling spring (Aleksandrovskiy and

Zaharova 2018). In the Pasternak foundation, a shear layer is introduced to produce coupling of the vertical springs. The resulting foundation reaction equation is:

$$q(x) = kw(x) - k_1 \frac{d^2w(x)}{dx^2} \quad (2)$$

while k_1 is the second parameter of the foundation.

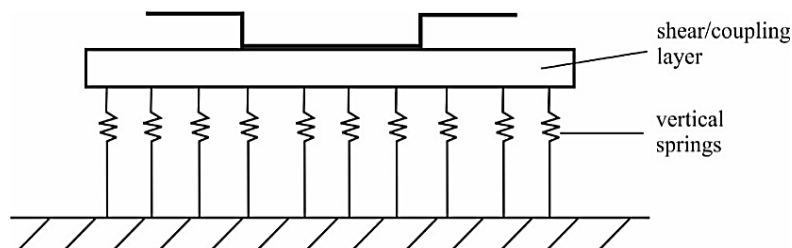


Fig. 2. Pasternak foundation model.

1.2. Previous studies

Ike (2018a) used finite Fourier sine transform method to determine the free transverse vibration frequencies of simply supported thin beam on Winkler foundations. The sinusoidal kernel function of the transformation satisfies the Dirichlet boundary conditions and yielded exact solutions to the eigenvalue problem. The work however did not consider other BCs, and was limited only to simply supported end conditions. Ike (2018b) used point collocation method for the analysis of static bending problems of Euler-Bernoulli beam on Winkler foundation (EBBoWF) but did not consider vibration problems. Ike and Elzaki (2023) explored the Elzaki transform method for the free vibration analysis of thin beams under various BCs, but they did not consider EBBoWF. Ike (2024b) used Stodolla-Vianello iteration method (SVIM) in novel ways for the free transverse vibration solutions of EBBoWF for simply supported BCs. The exact shape functions used in the SVIM gave exact eigenfrequencies for the n th iteration. The work failed to consider other end support conditions for the vibrating thin BoEF problem.

Coşkun (2003) studied the response of an elastic beam on two-dimensional (2D) tensionless Pasternak

foundation which is under a central point sinusoidal load. Erdogan and Coşkun (2020) used semi-analytical and finite element methods to study the free vibrations of a stepped beam on an elastic foundation. Adomian Decomposition Method (ADM) was the semi-analytical method adopted in their work and the finite element solutions were obtained via finite element computational software SAP 2000. Their work considered different support conditions and found favourable comparisons between the ADM results and SAP 2000 results.

Zhou (1993) developed a general solution for transverse vibration analysis of thin beams on variable Winkler elastic foundation. The dynamic response of the beams obtained in the work by considering the foundation reaction as an external force acting on the beam. The four unknown constants in the solution were found using the boundary conditions of the beam. The resulting integration was evaluated approximately using the trapezoidal rule. The frequency equation is a determinant with order that equals the number of discrete nodes. The mode shape functions were expressed as series of analytical functions. Accurate solutions were found using few discrete nodes in the trapezoidal quadrature and the study gave good convergence at lower computation effort.

Kanwal et al. (2024) used Galerkin finite element method (GFEM) and the separation of variables technique to find the eigenfrequencies and modal shapes of beams that are elastically constrained and resting on elastic foundations. They considered Winkler, Pasternak and Hetenyi foundation models and non-classical boundary conditions. They investigated the effects of flexural rigidity, modulus of elasticity and foundation parameters on the free vibration frequencies of beams modeled/idealized using Euler-Bernoulli, Timoshenko, Rayleigh and shear beam models. The GFEM was shown to converge to the closed form solution without shear locking in the stiffness element. They further found that the second foundation parameter in the form of shear layer and foundation constants led to an increase in natural vibration frequencies.

Al-Azzawi and Daud (2020) studied the natural transverse vibrations of non-prismatic, non-homogeneous beam resting on variable Winkler elastic foundations using the finite difference (FD) and finite element methods (FEMs). They considered different end support conditions for the beam. The FD method used in their work demands lesser computing rigour, compared to other methods and is utilized when there is no exact solution to the governing differential equation (GDE), or when closed form solutions are difficult to obtain. In the FD method, the GDE is expressed in finite difference form as algebraic expressions in terms of grid values of the unknowns. For the study, the FD form of the GDE reduced to an algebraic eigenvalue problem which was solved for nontrivial cases for the eigenvalues and hence the natural frequencies. The results showed favourable comparison with finite element method results obtained using ABAQUS.

Kacer et al. (2011) studied the natural transverse vibration analysis of Euler-Bernoulli beam on a variable Winkler foundation using the Differential Transform Method (DTM). Their work considered the Winkler parameter to vary along the beam's major axis, and considered cases of constant, linear and quadratic variations of the Winkler foundation parameter. The study considered three cases of boundaries, namely: simply supported at both ends, clamped at both ends, and cantilevered (clamped at one end and free at the other end). The GDE was solved using DTM which is a transformation technique relying on truncated Taylor series expansions and yielded high accuracy. Recently, Olotu et al. (2021) used DTM for the free vibration analysis of non-prismatic Rayleigh beams on Winkler foundations with varying foundation stiffness. Balkaya et al. (2009) also utilized DTM for the free vibration analysis of thin BoEFs.

Agboola and Gbadeyan (2013) used DTM to obtain accurate free vibration frequencies of thin beams, but they did not consider solutions for thin beam on elastic foundation. Adair et al. (2018) utilized the Variational Iteration Method (VIM) for the natural transverse vibration analysis of Timoshenko beams resting on two-parameter elastic foundation. The VIM is a powerful semi-analytical method for solving linear and nonlinear differential equations, including ordinary and partial differential equations. Their work produced a sequence of approximate solutions which can converge to the exact solution. VIM uses correction functional with Lagrange multipliers

to iteratively refine the solution. Yaylı et al. (2014) presented an efficient analytical method for the free vibration analysis of a beam on elastic foundation for non-classical boundary conditions of elastically restrained ends. Franciosi and Masi (1993) integrated the free vibration studies of foundation beams resting on bi-parametric elastic foundations.

Chen (2000) used the differential quadrature element method (DQEM) for the free transverse vibration analysis of beam resting on an elastic foundation. The work combined the ideas of differential quadrature method (DQM) and finite element method (FEM) to solve the governing differential equations of the problem. The DQM uses a numerical method for approximate expressions of derivatives at a point on the solution domain using a weighted form of the function's values at the neighbouring points while the FEM provides a numerical approach for solving the resulting differential equations by discretizing the domain into smaller elements. Accurate eigenfrequency results for various classical BCs were obtained by Chen (2000). Mutman and Coşkun (2013) used Homotopy Perturbation Method (HPM) to develop accurate free vibration solutions for non-prismatic Euler-Bernoulli beams on elastic foundation.

Rahbar-Ranji and Shahbazzabar (2017) used Legendre polynomials and Rayleigh-Ritz method for the eigenfrequencies analyses of beams on Pasternak foundation; and obtained accurate results. Ike (2025a) studied the static flexural analysis of thick beam on bi-parametric elastic foundation, but did not consider vibration analysis. Motaghan et al. (2018) developed a novel Fourier series solution for the natural transverse vibration of non-prismatic beams resting on variable elastic foundation. Soltani and Asgarian (2019) used a new hybrid approach for the free vibration analysis of axially functionally graded (FG) non-prismatic variable cross-section thin beams resting on uniform Winkler-Pasternak foundation. Rao and Raju (2002) developed accurate closed form eigenfrequency solutions for the free transverse vibration of thin beams on two-parameter elastic foundation. Ghannadial and Mofid (2015) presented analytical solutions for freely vibrating elastically restrained Timoshenko beam on variable Winkler foundation. A recent study by Ike (2025b) explored the application of Ritz variational method for the natural transverse harmonic vibration solutions of thin beam resting on biparametric elastic foundations. Satisfactory natural frequencies were obtained for various boundary conditions.

A recent innovative study by Ike (2025a) applied Stodola-Vianello successive iteration method to analyze natural vibration frequencies of thick beams based on Shimpi's shear deformable model. Closed form solutions were obtained in the study but the effect of elastic foundations on the thick beam vibration was not investigated.

Another recent work by Ike (2026) applied the generalized integral transformed method to analyze the natural vibrations of thin beam on two-parameter Filonenko-Borodich elastic foundations. Exact analytical solutions were obtained in the study, which considered various end support conditions, illustrating the accuracy and versatility of the GITM. The frequency results were validated using favorable comparisons with previous results based on other solution methods.

Uzun et al. (2023) investigated the dynamics of functionally graded nano-sized beams supported on Winkler elastic foundations for different boundary conditions; but did not consider the use of Elzaki integral transformations for solving the resulting boundary value problem. Civalek and Demir (2011, 2022) studied bending and buckling beams but did not consider dynamic analysis.

Akgoz et al. (2016) investigated the static bending of beams on an elastic foundation using discrete singular convolution (DSC), but did not consider the effect of elastic foundation and restrained the work to static bending. Akgoz and Civalek (2015) studied the flexural analysis of functional graded micro beams on Winkler elastic foundations using strain gradient elasticity method. Their work however did not consider natural vibration analysis and did not use ETM.

This work explores Elzaki transform method (ETM) for the free harmonic transverse vibration analysis of Euler-Bernoulli beams resting on Winkler foundation. This work aims to apply the ETM in a step by step systematic manner to develop analytical solutions for beam-foundation vibration and illustrate the effectiveness of the ETM for each analysis. The ETM has been satisfactorily used previously by Ike (2020) and Elzaki (2023) and by Elzaki (2011, 2012a, 2012b, 2012c) and also by Elzaki and Elzaki (2011) for solving differential equations. The ETM is adopted in this work because:

- It is efficient for both linear and nonlinear problems, and can be used for partial differential equations (PDEs).
- It reduces the computation effort, compared to rigorous methods for solving differential equations.
- It is applicable to diverse problems described using differential equations in engineering physics.

1.3. Novelty of the work

The novelty of the ETM lies in its ability to transform complex differential equations that govern problems into simple, solvable algebraic equations in the Elzaki space. The novel aspects of this study consists in

- The transformation of the governing differential equations of beam foundation vibration into simple solvable algebraic equations in the Elzaki space.
- The use of step by step, systematic procedure to solve the resulting algebraic equations for the classical BCs considered.
- ETM gives exact analytical solutions without the need for the common assumptions of linearization, perturbation or discretization.
- ETM directly considers the initial conditions of the problem and avoids some of the more complex convolution or expansion processes encountered in other transform methods.

2. Theoretical Framework

2.1. Governing equation

The governing equation for the transverse vibration of thin Euler-Bernoulli beam on Winkler foundation is the non-homogeneous partial differential equation (Ike 2018a, 2026):

$$EI \frac{\partial^4 w}{\partial x^4} + kw + \rho A \frac{\partial^2 w}{\partial t^2} = q(x, t) \quad (3)$$

where $w(x, t)$ is the transverse deflection, k is the Winkler foundation parameter, x is the longitudinal coordinate, t is the time, ρ is the mass density, A is the cross-sectional area, $q(x, t)$ is the applied load, E is the modulus of elasticity, I is the moment of inertia.

For free vibration,

$$q(x, t) = 0 \quad (4)$$

Hence, the governing differential equation (GDE) of free vibration is the homogeneous equation

$$EI \frac{\partial^4 w}{\partial x^4} + kw + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \quad (5)$$

For harmonic vibrations,

$$w(x, t) = W(x) \sin(\omega_n t + \phi) \quad (6)$$

where ω_n is the natural frequency, and $W(x)$ is the modal displacement function.

$$\frac{\partial^2 w}{\partial t^2} = -\omega_n^2 W(x) \sin(\omega_n t + \phi) \quad (7)$$

$$(EI W^{iv}(x) + kW(x) - \rho A \omega_n^2 W(x)) \sin(\omega_n t + \phi) = 0 \quad (8)$$

$$EI W^{iv}(x) + (k - \rho A \omega_n^2) W(x) = 0 \quad (9)$$

$$W^{iv}(x) + \left(\frac{k - \rho A \omega_n^2}{EI} \right) W(x) = 0 \quad (10)$$

$$W^{iv}(x) + (4\beta^4 - \Omega_n^4) W(x) = 0 \quad (11)$$

$$4\beta^4 = \frac{k}{EI}; \Omega_n^4 = \frac{\rho A \omega_n^2}{EI} \quad (12)$$

Let

$$\alpha_n^4 = \Omega_n^4 - 4\beta^4 \quad (13)$$

$$W^{iv}(x) - \alpha_n^4 W(x) = 0 \quad (14)$$

2.2. Elzaki transform method and its properties

The Elzaki transform is an integral transform which was derived from the Fourier integral by Elzaki (2011). The purpose was to simplify the process of solving linear ordinary and partial differential equations with constant or variable parameters. It is now a significant tool in applied mathematics, converting complex differential equations into easier algebraic equations.

The Elzaki transform of a function $f(x)$ is defined as: (Elzaki 2011, 2012a, 2012b, 2012c)

$$Tf(x) = u \int_0^\infty f(x) e^{-x/u} dx = u \lim_{\alpha \rightarrow \infty} \int_0^\alpha f(x) e^{-x/u} dx = \int_0^\infty f(x) e^{-x/u} dx = F(u) \quad (15)$$

where T is the Elzaki transform operator, u is the Elzaki transform parameter.

The Elzaki transformation transforms a function dependent on the physical domain x to another function dependent on the Elzaki transform space (domain) parameter u .

For a function $f(x, t)$ of two independent variables, x and time t , the Elzaki transform can be defined with respect to the parameters x and t as:

$$T_x f(x, t) = T(u, t) = u \int_0^\infty f(x, t) e^{-x/u} dx = u \lim_{\alpha \rightarrow \infty} \int_0^\alpha f(x, t) e^{-x/u} dx \quad (16)$$

$$T_t f(x, t) = T(x, v) = F(x, v) = v \int_0^\infty f(x, t) e^{-t/v} dt = v \lim_{\beta \rightarrow \infty} \int_0^\beta f(x, t) e^{-t/v} dt \quad (17)$$

where T_x denotes Elzaki transform with respect to x , and T_t is Elzaki transform with respect to t .

The Elzaki transform method simplifies the solution of initial value problems by transformation of function derivatives into simple algebraic expressions. The Elzaki transforms of derivatives are obtained using integration by parts as:

$$T f'(x) = u \int_0^\infty \frac{df(x)}{dx} e^{-x/u} dx \quad (18)$$

$$T f'(x) = u [f(x) e^{-x/u}]_0^\infty - \int_0^\infty f(x) \left(\frac{-e^{-x/u}}{u} \right) dx \quad (19)$$

$$T f'(x) = u \left(f(\infty) e^{-\infty} - f(0) e^0 + \frac{1}{u} \int_0^\infty f(x) e^{-x/u} dx \right) \quad (20)$$

$$T f'(x) = \int_0^\infty f(x) e^{-x/u} dx - u f(0) \quad (21)$$

$$T f'(x) = \frac{T f(x)}{u} - u f(0) = \frac{F(u)}{u} - u f(0) \quad (22)$$

$$\text{Let } g(x) = f'(x) \quad (23)$$

$$T g'(x) = T f''(x) = \frac{1}{u} T g(x) - u g(0) \quad (24)$$

$$T f''(x) = \frac{1}{u} \left(\frac{F(u)}{u} - u f(0) \right) - u f'(0) \quad (25)$$

$$T f''(x) = \frac{F(u)}{u^2} - f(0) - u f'(0) \quad (26)$$

$$\text{Let } g(x) = f''(x), g'(x) = f'''(x)$$

$$T g'(x) = T f'''(x) = \frac{1}{u} (T g(x)) - u g(0) \quad (27)$$

$$T f'''(x) = \frac{1}{u} (T f''(x)) - u f''(0) \quad (28)$$

$$T f'''(x) = \frac{1}{u} \left(\frac{F(u)}{u^2} - f(0) - u f'(0) \right) - u f''(0) \quad (29)$$

$$T f'''(x) = \frac{F(u)}{u^3} - \frac{1}{u} f(0) - f'(0) - u f''(0) \quad (30)$$

Similarly,

$$T f^{iv}(x) = \frac{1}{u} (T f'''(x)) - u f'''(0) \quad (31)$$

$$T f^{iv}(x) = \frac{1}{u} \left(\frac{F(u)}{u^3} - \frac{1}{u} f(0) - f'(0) - u f''(0) \right) - u f'''(0) \quad (32)$$

$$T f^{iv}(x) = \frac{F(u)}{u^4} - \frac{f(0)}{u^2} - \frac{1}{u} f'(0) - f''(0) - u f'''(0) \quad (33)$$

2.3. Flowchart for implementing the ETM

1. Define the governing differential equation (GDE) and initial conditions.
2. Apply the Elzaki transformation to the GDE by multiplying the GDE by $u e^{-x/u}$ and integrating with respect to x from $x = 0$ to $x = \infty$. The problem is thus converted from the space (problem) domain to the Elzaki domain, u .
3. Substitute initial conditions by incorporating the given initial conditions into the newly transformed algebraic equation in the Elzaki transform domain.
4. Rearrange and solve algebraically to obtain a solution in the Elzaki domain.
5. Apply the inverse Elzaki transform by using standard inverse Elzaki transform formulas or by using partial fraction decomposition techniques to simplify complex algebraic problems to simpler ones with existing inverses. Thus, the solution in the Elzaki domain is found in the problem domain.
6. Final solution: The final solution gives the analytical solution to the GDE subject to the initial conditions.

3. Methodology

Applying the Elzaki transformation to the governing equation yields:

$$u \int_0^\infty \left(\frac{d^4 W(x)}{dx^4} - \alpha_n^4 W(x) \right) e^{-x/u} dx = 0 \quad (34)$$

wherein, u is the Elzaki transform parameter.

Using the linear properties of the Elzaki transformation, gives:

$$u \int_0^\infty \frac{d^4 W(x)}{dx^4} e^{-x/u} dx - u \alpha_n^4 \int_0^\infty W(x) e^{-x/u} dx = 0 \quad (35)$$

Hence, simplifying Eq. (35) gives:

$$\frac{W(u)}{u^4} - \frac{W(0)}{u^2} - \frac{W'(0)}{u} - W''(0) - u W'''(0) - \alpha^4 W(u) = 0 \quad (36)$$

where

$$W(u) = u \int_0^\infty W(x) e^{-x/u} dx \quad (37)$$

Hence, further simplification gives:

$$\frac{W(u)}{u^4} - \alpha_n^4 W(u) = \frac{W(0)}{u^2} + \frac{W'(0)}{u} + W''(0) + u W'''(0) \quad (38)$$

Further simplification gives:

$$\frac{W(u) - \alpha_n^4 u^4 W(u)}{u^4} = \frac{W(0)}{u^2} + \frac{W'(0)}{u} + W''(0) + u W'''(0) \quad (39)$$

Hence,

$$W(u) \left(\frac{1-\alpha_n^4 u^4}{u^4} \right) = \frac{W(0)}{u^2} + \frac{W'(0)}{u} + W''(0) + uW'''(0) \quad (40)$$

Then,

$$W(u) = W(0) \left(\frac{u^2}{1-\alpha_n^4 u^4} \right) + W'(0) \left(\frac{u^3}{1-\alpha_n^4 u^4} \right) + W''(0) \left(\frac{u^4}{1-\alpha_n^4 u^4} \right) + W'''(0) \left(\frac{u^5}{1-\alpha_n^4 u^4} \right) \quad (41)$$

Inverting,

$$W(x) = T^{-1}W(u) \quad (42)$$

where T^{-1} is the inverse Elzaki transform of $W(u)$. Then,

$$W(x) = T^{-1} \left(W(0) \left(\frac{u^2}{1-\alpha_n^4 u^4} \right) + W'(0) \left(\frac{u^3}{1-\alpha_n^4 u^4} \right) + W''(0) \left(\frac{u^4}{1-\alpha_n^4 u^4} \right) + W'''(0) \left(\frac{u^5}{1-\alpha_n^4 u^4} \right) \right) \quad (43)$$

Using the linearity properties of the inverse transform gives:

$$W(x) = W(0)T^{-1} \left(\frac{u^2}{1-\alpha_n^4 u^4} \right) + W'(0)T^{-1} \left(\frac{u^3}{1-\alpha_n^4 u^4} \right) + W''(0)T^{-1} \left(\frac{u^4}{1-\alpha_n^4 u^4} \right) + W'''(0)T^{-1} \left(\frac{u^5}{1-\alpha_n^4 u^4} \right) \quad (44)$$

By partial fraction decomposition methods and the catalogue of inverse Elzaki transforms, the inverse Elzaki transforms are found by expressing the required inverses using partial fractions.

Let A and B be constants, then,

$$\frac{u^2}{1-\alpha_n^4 u^4} = \frac{Au^2}{1+\alpha_n^2 u^2} + \frac{Bu^2}{1-\alpha_n^2 u^2} \quad (45)$$

$$\frac{u^2}{1-\alpha_n^4 u^4} = \frac{u^2 A(1-\alpha_n^2 u^2) + Bu^2(1+\alpha_n^2 u^2)}{1-\alpha_n^4 u^4} \quad (46)$$

$$\frac{u^2}{1-\alpha_n^4 u^4} = \frac{u^2(A+B) + u^4 \alpha_n^2(B-A)}{1-\alpha_n^4 u^4} \quad (47)$$

Hence, $A + B = 1$ and $B - A = 0$

Solving,

$$A = B = \frac{1}{2} \quad (48)$$

$$\frac{u^2}{1-\alpha_n^4 u^4} = \frac{1}{2} \left(\frac{u^2}{1+\alpha_n^2 u^2} \right) + \frac{1}{2} \left(\frac{u^2}{1-\alpha_n^2 u^2} \right) \quad (49)$$

$$T^{-1} \left(\frac{u^2}{1-\alpha_n^4 u^4} \right) = \cosh(\alpha_n x) \quad (50)$$

$$T^{-1} \left(\frac{u^2}{1+\alpha_n^2 u^2} \right) = \cos(\alpha_n x) \quad (51)$$

Hence,

$$T^{-1} \left(\frac{u^2}{1-\alpha_n^4 u^4} \right) = \frac{1}{2} (\cosh(\alpha_n x) + \cos(\alpha_n x)) \quad (52)$$

Similarly,

$$\frac{u^3}{1-\alpha_n^4 u^4} = \frac{A\alpha_n u^3}{1-\alpha_n^2 u^2} + \frac{B\alpha_n u^3}{1+\alpha_n^2 u^2} \quad (53)$$

$$\frac{u^3}{1-\alpha_n^4 u^4} = \frac{A\alpha_n u^3(1+\alpha_n^2 u^2) + B\alpha_n u^3(1-\alpha_n^2 u^2)}{1-\alpha_n^4 u^4} \quad (54)$$

$$\frac{u^3}{1-\alpha_n^4 u^4} = \frac{A\alpha_n u^3 + B\alpha_n u^3 + (A\alpha_n u^3 - B\alpha_n u^3)\alpha_n^2 u^2}{1-\alpha_n^4 u^4} \quad (55)$$

$$u^3 \equiv (A+B)\alpha_n u^3 + (A-B)\alpha_n^3 u^5 \quad (56)$$

$$A-B=0; (A+B)\alpha_n=1; A=B=\frac{1}{2\alpha_n} \quad (57)$$

$$\frac{u^3}{1-\alpha_n^4 u^4} = \frac{1}{2\alpha_n} \left(\frac{\alpha_n u^3}{1-u^2 \alpha_n^2} + \frac{\alpha_n u^3}{1+\alpha_n^2 u^2} \right) \quad (58)$$

$$T^{-1} \left(\frac{u^3}{1-\alpha_n^4 u^4} \right) = \frac{1}{2\alpha_n} \left(T^{-1} \left(\frac{\alpha_n u^3}{1-u^2 \alpha_n^2} \right) + T^{-1} \left(\frac{\alpha_n u^3}{1+\alpha_n^2 u^2} \right) \right) \quad (59)$$

$$T^{-1} \frac{u^3}{1-\alpha_n^4 u^4} = \frac{1}{2\alpha_n} (\sinh(\alpha_n x) + \sin(\alpha_n x)) \quad (60)$$

Also,

$$\frac{u^4}{1-\alpha_n^4 u^4} = \frac{Au^2}{1-\alpha_n^2 u^2} + \frac{Bu^2}{1+\alpha_n^2 u^2} \quad (61)$$

$$\frac{u^4}{1-\alpha_n^4 u^4} = \frac{Au^2(1+\alpha_n^2 u^2) + Bu^2(1-\alpha_n^2 u^2)}{1-\alpha_n^4 u^4} \quad (62)$$

$$u^4 = (A+B)u^2 + \alpha_n^2 u^4(A-B) \quad (63)$$

$$A+B=0; (A-B)\alpha_n^2=1; A=-B; A-B=\frac{1}{\alpha_n^2}=2A; \quad (64)$$

$$A=\frac{1}{2\alpha_n^2}, B=-\frac{1}{2\alpha_n^2}$$

Hence,

$$\frac{u^4}{1-\alpha_n^4 u^4} = \frac{1}{2\alpha_n^2} \left(\frac{u^2}{1-\alpha_n^2 u^2} - \frac{u^2}{1+\alpha_n^2 u^2} \right) \quad (65)$$

$$T^{-1} \left(\frac{u^4}{1-\alpha_n^4 u^4} \right) = \frac{1}{2\alpha_n^2} \left(T^{-1} \left(\frac{u^2}{1-\alpha_n^2 u^2} \right) - T^{-1} \left(\frac{u^2}{1+\alpha_n^2 u^2} \right) \right) \quad (66)$$

$$T^{-1} \left(\frac{u^4}{1-\alpha_n^4 u^4} \right) = \frac{1}{2\alpha_n^2} (\cosh(\alpha_n x) - \cos(\alpha_n x)) \quad (67)$$

Similarly,

$$\frac{u^5}{1-\alpha_n^4 u^4} = \frac{A\alpha_n u^3}{1-\alpha_n^2 u^2} + \frac{B\alpha_n u^3}{1+\alpha_n^2 u^2} \quad (68)$$

$$\frac{u^5}{1-\alpha_n^4 u^4} = \frac{A\alpha_n u^3(1+\alpha_n^2 u^2) + B\alpha_n u^3(1-\alpha_n^2 u^2)}{1-\alpha_n^4 u^4} \quad (69)$$

$$u^5 \equiv (A+B)\alpha_n u^3 + \alpha_n^3 u^5(A-B) \quad (70)$$

Comparing like terms

$$A+B=0; (A-B)\alpha_n^3=1; A=-B; 2A\alpha_n^3=1; \quad (71)$$

$$A=\frac{1}{2\alpha_n^3}, B=-\frac{1}{2\alpha_n^3}$$

$$\frac{u^5}{1-\alpha_n^4 u^4} = \frac{1}{2\alpha_n^3} \left(\frac{\alpha_n u^3}{1-\alpha_n^2 u^2} - \frac{\alpha_n u^3}{1+\alpha_n^2 u^2} \right) \quad (72)$$

$$T^{-1} \left(\frac{u^5}{1-\alpha_n^4 u^4} \right) = \frac{1}{2\alpha_n^3} \left(T^{-1} \left(\frac{\alpha_n u^3}{1-\alpha_n^2 u^2} \right) - T^{-1} \left(\frac{\alpha_n u^3}{1+\alpha_n^2 u^2} \right) \right) \quad (73)$$

$$T^{-1} \left(\frac{u^5}{1-\alpha_n^4 u^4} \right) = \frac{1}{2\alpha_n^3} (\sinh(\alpha_n x) - \sin(\alpha_n x)) \quad (74)$$

Hence,

$$W(x) = W(0) \left(\frac{\cosh \alpha_n x + \cos \alpha_n x}{2} \right) + W'(0) \left(\frac{\sinh \alpha_n x + \sin \alpha_n x}{2\alpha_n} \right) + W''(0) \left(\frac{\cosh \alpha_n x - \cos \alpha_n x}{2\alpha_n^2} \right) + W'''(0) \left(\frac{\sinh \alpha_n x - \sin \alpha_n x}{2\alpha_n^3} \right) \quad (75)$$

Differentiating,

$$W'(x) = W(0)\alpha_n \left(\frac{\sinh \alpha_n x - \sin \alpha_n x}{2} \right) + W'(0) \left(\frac{\cosh \alpha_n x + \cos \alpha_n x}{2} \right) + W''(0) \left(\frac{\sinh \alpha_n x + \sin \alpha_n x}{2\alpha_n} \right) + W'''(0) \left(\frac{\cosh \alpha_n x - \cos \alpha_n x}{2\alpha_n^2} \right) \quad (76)$$

$$W''(x) = W(0)\alpha_n^2 \left(\frac{\cosh \alpha_n x - \cos \alpha_n x}{2} \right) + W'(0)\alpha_n \left(\frac{\sinh \alpha_n x - \sin \alpha_n x}{2} \right) + W''(0) \left(\frac{\cosh \alpha_n x + \cos \alpha_n x}{2} \right) + W'''(0) \left(\frac{\sinh \alpha_n x + \sin \alpha_n x}{2\alpha_n} \right) \quad (77)$$

$$W'''(x) = W(0)\alpha_n^3 \left(\frac{\sinh \alpha_n x + \sin \alpha_n x}{2} \right) + W'(0)\alpha_n^2 \left(\frac{\cosh \alpha_n x - \cos \alpha_n x}{2} \right) + W''(0)\alpha_n \left(\frac{\sinh \alpha_n x - \sin \alpha_n x}{2} \right) + W'''(0) \left(\frac{\cosh \alpha_n x + \cos \alpha_n x}{2} \right) \quad (78)$$

4. Results and Discussion

4.1. Results

4.1.1. Results for simply supported EBBowF

The results are presented for simply supported EBBowF shown in Fig. 3.

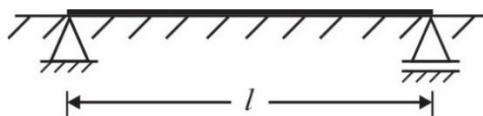


Fig. 3. Simply supported EBBowF.

The BCs for the simply supported EBBowF shown in Fig. 3 are:

$$W(0) = W(l) = 0, \quad W''(0) = W''(l) = 0 \quad (79)$$

Hence,

$$\begin{pmatrix} \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) & \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2\alpha_n^3} \right) \\ \alpha_n \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2} \right) & \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) \end{pmatrix} \begin{pmatrix} W'(0) \\ W'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (80)$$

For nontrivial solutions, the characteristic equation is:

$$\alpha_n \begin{pmatrix} \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) & \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2\alpha_n^3} \right) \\ \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2} \right) & \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) \end{pmatrix} = 0 \quad (81)$$

Expanding,

$$\left(\frac{(\sinh \alpha_n l + \sin \alpha_n l)^2}{4\alpha_n^2} \right) - \left(\frac{(\sinh \alpha_n l - \sin \alpha_n l)^2}{4\alpha_n^2} \right) = 0 \quad (82)$$

$$\frac{1}{4\alpha_n^2} [(\sinh^2 \alpha_n l + 2 \sinh \alpha_n l \sin \alpha_n l + \sin^2 \alpha_n l) - (\sinh^2 \alpha_n l - 2 \sinh \alpha_n l \sin \alpha_n l + \sin^2 \alpha_n l)] = 0 \quad (83)$$

Simplifying,

$$\frac{1}{4\alpha_n^2} 4 \sinh \alpha_n l \sin \alpha_n l = 0 \quad (84)$$

or,

$$\frac{\sinh \alpha_n l \sin \alpha_n l}{\alpha_n^2} = 0 \quad (85)$$

Hence,

$$\sin \alpha_n l = 0; \text{ or, } \sinh \alpha_n l = 0 \quad (86)$$

$$\alpha_n l = \sin^{-1} 0 = n\pi \quad (87)$$

$$\alpha_n = \frac{n\pi}{l} \quad (88)$$

$$\alpha_n^4 = -4\beta^4 + \Omega_n^4 = \left(\frac{n\pi}{l} \right)^4 \quad (89)$$

Hence,

$$\Omega_n^4 = 4\beta^4 + \left(\frac{n\pi}{l} \right)^4 = \frac{\rho A \omega_n^2}{EI} \quad (90)$$

Thus,

$$\omega_n^2 = \frac{EI}{\rho A} \left(4\beta^4 + \left(\frac{n\pi}{l} \right)^4 \right) = \frac{EI}{\rho A} \left(\frac{(n\pi)^4}{l^4} + 4\beta^4 \right) \quad (91)$$

$$\omega_n^2 = \frac{EI}{\rho A} \left(\frac{(n\pi)^4 + 4\beta^4 l^4}{l^4} \right) = \frac{EI}{\rho A l^4} ((n\pi)^4 + 4\beta^4 l^4) = \frac{EI}{\rho A l^4} ((n\pi)^4 + K_0) \quad (92)$$

where,

$$K_0 = 4\beta^4 l^4 \quad (92a)$$

K_0 is the beam foundation parameter.

Hence,

$$\omega_n = \frac{1}{l^2} \sqrt{\frac{EI}{\rho A} ((n\pi)^4 + 4\beta^4 l^4)^{1/2}} \quad (93)$$

or,

$$\omega_n = \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}} \lambda_n^2 \quad (94)$$

where,

$$\lambda_n^2 = ((n\pi)^4 + 4\beta^4 l^4)^{1/2} = ((n\pi)^4 + K_0)^{1/2} \quad (95)$$

The values of λ_n are calculated for $n = 1, 2, 3, 4, 5$ and for $4\beta^4 l^4 = 1, 10, 100, 1000, 10,000$ and presented in Table 1 along with previous solutions by Rahbar-Ranji and Shahbazzabar (2017), Zhou (1993) and Ike (2023).

Table 1. Dimensionless frequency parameters λ_n for the first five modes of simply supported EBBowF.

$K_0 = 4\beta^4 l^4$	λ_1					λ_2				
	Present study	% Difference between present results and Ike (2023, 2024), Kacer et al. (2011)	Ike (2023, 2024b), Kacer et al. (2011)	Rahbar-Ranji and Shahbazzabbar (2017)	Zhou (1993)	Present study	% Difference between present results and Ike (2023, 2024), Kacer et al. (2011)	Ike (2023, 2024b), Kacer et al. (2011)	Rahbar-Ranji and Shahbazzabbar (2017)	Zhou (1993)
1	3.1496	(0%)	3.1496	3.1496	-	6.2842	(0%)	6.2842	6.2843	-
10	3.2193	(0%)	3.2193	3.2193	3.2200	6.2932	(0%)	6.2932	6.2932	6.2930
100	3.7484	(0%)	3.7484	3.7484	3.7480	6.3816	(0%)	6.3816	6.3816	6.3820
1000	5.7556	(0%)	5.7556	5.7556	5.7550	7.1121	(0%)	7.1121	7.1121	7.1120
10,000	10.0243	(0%)	10.0243	10.0243	-	10.3687	(0%)	10.3687	10.3687	-

$K_0 = 4\beta^4 l^4$	λ_3					λ_4				
	Present study	% Difference between present results and Ike (2023, 2024), Kacer et al. (2011)	Ike (2023, 2024b), Kacer et al. (2011)	Rahbar-Ranji and Shahbazzabbar (2017)	Zhou (1993)	Present study	% Difference between present results and Ike (2023, 2024), Kacer et al. (2011)	Ike (2023, 2024b), Kacer et al. (2011)	Rahbar-Ranji and Shahbazzabbar (2017)	Zhou (1993)
1	9.4251	(0%)	9.4251	9.4251	-	12.5665	(0%)	12.5665	12.5665	-
10	9.4278	(0%)	9.4278	9.4277	9.4270	12.5676	(0%)	12.5676	12.5676	12.5680
100	9.4545	(0%)	9.4545	9.4545	9.4540	12.5789	(0%)	12.5789	12.5790	12.5790
1000	9.7102	(0%)	9.7102	9.7102	9.7100	12.6905	(0%)	12.6905	12.6905	12.6900
10,000	11.5652	(0%)	11.5652	11.5652	-	13.6716	(0%)	13.6716	13.6716	-

$K_0 = 4\beta^4 l^4$	λ_5				
	Present study	% Difference between present results and Ike (2023, 2024), Kacer et al. (2011)	Ike (2023, 2024b), Kacer et al. (2011)	Rahbar-Ranji and Shahbazzabbar (2017)	Zhou (1993)
1	15.7080	(0%)	15.7080	15.7080	-
10	15.7086	(0%)	15.7086	15.7086	15.7090
100	15.7144	(0%)	15.7144	15.7144	15.7150
1000	15.7721	(0%)	15.7721	15.7721	-
10,000	16.3167	(0%)	16.3167	16.3167	-

4.1.2. EBBowF with clamped ends $x = 0, x = l$

The natural vibration of EBBowF with both left and right ends clamped as shown in Fig. 4 is considered.

**Fig. 4.** EBBowF with clamped ends at $x = 0$ and $x = l$.

The BCs are $W(0) = W'(0) = 0$; $W(l) = W'(l) = 0$
Hence,

$$\begin{pmatrix} \left(\frac{\cosh \alpha_n l - \cos \alpha_n l}{2\alpha_n^2} \right) & \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2\alpha_n^3} \right) \\ \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) & \left(\frac{\cosh \alpha_n l - \cos \alpha_n l}{2\alpha_n^2} \right) \end{pmatrix} \begin{pmatrix} W''(0) \\ W'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (96)$$

For nontrivial solutions,

$$\begin{pmatrix} \left(\frac{\cosh \alpha_n l - \cos \alpha_n l}{2\alpha_n^2} \right) & \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2\alpha_n^3} \right) \\ \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) & \left(\frac{\cosh \alpha_n l - \cos \alpha_n l}{2\alpha_n^2} \right) \end{pmatrix} = 0 \quad (97)$$

Expanding,

$$\left(\frac{\cosh \alpha_n l - \cos \alpha_n l}{2\alpha_n^2} \right)^2 - \frac{1}{4\alpha_n^4} (\sinh^2 \alpha_n l - \sin^2 \alpha_n l) = 0 \quad (98)$$

Expanding,

$$\frac{1}{4\alpha_n^4} (\cosh^2 \alpha_n l - 2 \cosh \alpha_n l \cos \alpha_n l + \cos^2 \alpha_n l - \sinh^2 \alpha_n l + \sin^2 \alpha_n l) = 0 \quad (99)$$

$$\therefore \cosh^2 \alpha_n l - \sinh^2 \alpha_n l + \cos^2 \alpha_n l + \sin^2 \alpha_n l - 2 \cosh \alpha_n l \cos \alpha_n l = 0 \quad (100)$$

Using trigonometric identities,

$$\cosh^2 \alpha_n l - \sinh^2 \alpha_n l = 1; \cos^2 \alpha_n l + \sin^2 \alpha_n l = 1 \quad (101)$$

Eq. (100) simplifies to Eq. (102)

$$2 - 2 \cosh \alpha_n l \cos \alpha_n l = 0 \quad (102)$$

Hence,

$$\cosh \alpha_n l \cos \alpha_n l = 1 \quad (103)$$

The eigenvalues are the roots which are extracted using Newton Raphson's iteration. Eq. (103) is expressed in function form as:

$$f(x) = \cosh x \cos x - 1 = 0 \quad (104)$$

By differentiation,

$$f'(x) = \sinh x \cos x - \cosh x \sin x \quad (105)$$

The Newton-Raphson iterative relationship is the recurrence formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (106)$$

$$x_{n+1} = x_n - \frac{\cosh x_n \cos x_n - 1}{\sinh x_n \cos x_n - \cosh x_n \sin x_n} \quad (107)$$

For first iteration, $x_0 = 4.70$

• Iteration 1

$$x_1 = 4.70 - \frac{\cosh 4.7 \cos 4.7 - 1}{\sinh 4.7 \cos 4.7 - \cosh 4.7 \sin 4.7} = 4.730964$$

• Iteration 2

$$x_2 = 4.730964 - \frac{f(4.730964)}{f'(4.730964)} = 4.730042$$

• Iteration 3

$$x_3 = 4.730042 - \frac{f(4.730042)}{f'(4.730042)} = 4.730041$$

Table 2 shows the Newton-Raphsons iteration calculations presented in table format. The eigenvalues have converged to six decimal places since the next iteration yields.

The first positive non-trivial solution to Eq. (103) obtained using Newton-Raphson's iteration is

$$x_3 = 4.730041 \quad (108)$$

Table 2. Newton-Raphson's Iteration calculation for first natural frequencies from $f(x) = \cosh x \cos x - 1 = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	Error
0	4.70	-1.6811066	54.292922	4.730964	-1.6811066
1	4.730964	0.053270	57.750148	4.730042	0.053270
2	4.730042	7.235313×10^{-5}	57.645663	4.730041	7.235313×10^{-5}
3	4.730041	1.470753×10^{-5}	57.645549	4.730041	1.470753×10^{-5}
4	4.730041	1.470753×10^{-5}			1.470753×10^{-5}

Solving using Mathematica software, or Online Equation Solver of Wolfram Alpha, the first five eigenvalues of Eq. (103) are found as

$$\alpha_1 l = 4.73004; \alpha_2 l = 7.85321; \alpha_3 l = 10.9956$$

$$\alpha_4 l = 14.13717; \alpha_5 l = 17.27876 \quad (109)$$

Hence,

$$\alpha_n^4 = \Omega_n^4 - 4\beta^4 \quad (110)$$

Rearranging,

$$\Omega_n^4 = \alpha_n^4 + 4\beta^4 = \frac{\rho A \omega_n^2}{EI} \quad (111)$$

$$\omega_n^2 = \frac{EI}{\rho A} (\alpha_n^4 + 4\beta^4) \quad (112)$$

$$\omega_n^2 = \frac{EI}{\rho A l^4} (\alpha_n^4 l^4 + 4\beta^4 l^4) \quad (113)$$

$$\omega_n = \frac{1}{l^2} \sqrt{\frac{EI}{\rho A}} \lambda_n^2 \quad (114)$$

$$\lambda_n^2 = (\alpha_n^4 l^4 + 4\beta^4 l^4)^{1/2} = (\alpha_n^4 l^4 + K_0)^{1/2} \quad (115)$$

The values of λ_n are calculated for $n = 1, 2, 3, 4$ and for $4\beta^4 l^4 = 1, 10, 100, 1000, 10,000$ and presented in Table 3 along with previous results by Ike (2023), Kacer et al. (2011) and Balkaya et al. (2009). Table 4 shows the values of λ_n^2 for EBBWF with clamped boundaries for $\rho = A = l = E = 1, 4\beta^4 l^4 = 1$ together with previous results using Homotopy Perturbation method by Mutman and Coşkun (2013), Balkaya et al. (2009), Chen (2000) and Ike (2023).

4.1.3. EBBWF clamped at $x = l$ and simply supported at $x = 0$

The natural vibration of EBBWF with simple support at $x = 0$ and clamped at $x = l$ as shown in Fig. 5 is considered.



Fig. 5. Vibration of EBBWF clamped at $x = l$ and simply supported at $x = 0$.

Table 3. Natural vibration frequency parameters λ_n for the first four vibration modes of clamped-clamped beam on Winkler foundation.

$K_0 = 4\beta^4 l^4$	λ_1				λ_2			
	Present study	% Difference between present results and Ike (2023)	Ike (2023)	Balkaya et al. (2009), Kacer et al. (2011)	Present study	% Difference between present results and Ike (2023)	Ike (2023)	Balkaya et al. (2009), Kacer et al. (2011)
1	4.7324	(0%)	4.7324	4.7324	7.8537	(0%)	7.8537	7.8537
10	4.7535	(0%)	4.7535	4.7535	7.8584	(0%)	7.8584	7.8584
100	4.9504	(0%)	4.9504	4.9504	7.9043	(0%)	7.9043	7.9043
1000	6.2239	(0%)	6.2239	6.2239	8.3251	(0%)	8.3251	8.3251
10,000	10.1229	(0%)	10.1229	-	10.8392	(0%)	10.8392	-

$K_0 = 4\beta^4 l^4$	λ_3				λ_4			
	Present study	% Difference between present results and Ike (2023)	Ike (2023)	Balkaya et al. (2009), Kacer et al. (2011)	Present study	% Difference between present results and Ike (2023)	Ike (2023)	Balkaya et al. (2009), Kacer et al. (2011)
1	10.9958	(0%)	10.9958	10.9958	14.1373	(0%)	14.1373	14.1373
10	10.9975	(0%)	10.9975	10.9975	14.1381	(0%)	14.1381	14.1381
100	11.0144	(0%)	11.0144	11.0144	14.1460	(0%)	14.1460	14.1460
1000	11.1790	(0%)	11.1790	11.1790	14.2248	(0%)	14.2248	14.2248
10,000	12.5260	(0%)	12.5260	-	14.9493	(0%)	14.9493	14.9493

Table 4. Natural frequencies, ω_n for clamped-clamped EB beam resting on Winkler foundation for $\rho = A = I = E = l = 1$, $K_0 = 4\beta^4 l^4 = 1$, $\omega_n = \lambda_n^2$.

Method	ω_1	ω_2	ω_3	ω_4
Present method	22.3956	61.6810	120.9074	199.8621
Ike (2023)	22.3956	61.6810	120.9074	199.8621
HPM (Mutman and Coşkun 2013)	22.3956	61.6809	120.9080	199.8620
DTM (Balkaya et al. 2009)	22.3733	61.6728	120.9030	199.8590
ADM (Mutman and Coşkun 2013)	22.3956	61.6809	120.9080	199.8620
DQEM (Chen 2000)	22.3956	61.6811	120.9100	199.8850

The BCs are $W(0) = 0$, $W''(0) = 0$, $W(l) = 0$, $W'(l) = 0$.
Hence,

$$\begin{pmatrix} \frac{(\sinh \alpha_n l + \sin \alpha_n l)}{2\alpha_n} \\ \frac{(\cosh \alpha_n l + \cos \alpha_n l)}{2} \end{pmatrix} \begin{pmatrix} \frac{(\sinh \alpha_n l - \sin \alpha_n l)}{2\alpha_n^3} \\ \frac{(\cosh \alpha_n l - \cos \alpha_n l)}{2\alpha_n^2} \end{pmatrix} \begin{pmatrix} W'(0) \\ W'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (116)$$

For nontrivial solutions,

$$\begin{pmatrix} \frac{(\sinh \alpha_n l + \sin \alpha_n l)}{2\alpha_n} \\ \frac{(\cosh \alpha_n l + \cos \alpha_n l)}{2} \end{pmatrix} \begin{pmatrix} \frac{(\sinh \alpha_n l - \sin \alpha_n l)}{2\alpha_n^3} \\ \frac{(\cosh \alpha_n l - \cos \alpha_n l)}{2\alpha_n^2} \end{pmatrix} = 0 \quad (117)$$

Expanding,

$$\frac{1}{4\alpha_n^3} [(\sinh \alpha_n l + \sin \alpha_n l)(\cosh \alpha_n l - \cos \alpha_n l) - (\sinh \alpha_n l - \sin \alpha_n l)(\cosh \alpha_n l + \cos \alpha_n l)] = 0 \quad (118)$$

Hence,

$$\begin{aligned} &(\sinh \alpha_n l \cosh \alpha_n l + \sin \alpha_n l \cosh \alpha_n l - \cos \alpha_n l \sinh \alpha_n l - \sin \alpha_n l \cos \alpha_n l) - \\ &(\sinh \alpha_n l \cosh \alpha_n l - \sin \alpha_n l \cosh \alpha_n l + \sinh \alpha_n l \cos \alpha_n l - \sin \alpha_n l \cos \alpha_n l) = 0 \end{aligned} \quad (119)$$

Simplifying,

$$2 \sin \alpha_n l \cosh \alpha_n l - 2 \cos \alpha_n l \sinh \alpha_n l = 0 \quad (120)$$

$$\therefore \sin \alpha_n l \cosh \alpha_n l = \cos \alpha_n l \sinh \alpha_n l \quad (121)$$

$$\sin \alpha_n l = \cos \alpha_n l \frac{\sinh \alpha_n l}{\cosh \alpha_n l} \quad (122)$$

$$\frac{\sin \alpha_n l}{\cos \alpha_n l} = \frac{\sinh \alpha_n l}{\cosh \alpha_n l} \quad (123)$$

$$\tan \alpha_n l = \tanh \alpha_n l \quad (124)$$

The equation $f(x) = \tan x - \tanh x = 0$ has a trivial solution at $x = 0$, since $f(x = 0) = 0$, but its first non-trivial positive root is $x \approx 3.926602$. The first non-trivial positive root can be extracted using Newton-Raphson iteration method. The iteration scheme is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan x_n - \tanh x_n}{\sec^2 x_n - \operatorname{sech}^2 x_n} \quad (125)$$

The iteration can start at $x_0 = \frac{5\pi}{4} = 3.927$

Then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.927 - \frac{0.000988}{1.999393} = 3.926506$$

Similarly,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}; x_2 = 3.926506 - \frac{(-0.000192)}{1.998214} = 3.926602$$

Then,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 3.926602$$

The iteration has converged at $x = 3.926602$

The calculations are summarized in Table 5.

Table 5. Newton-Raphson iteration calculations for $f(x) = \tan x - \tanh x = 0$.

Iteration n	x_1	$f(x_n)$	$f'(x_n)$	x_{n+1}	Error
0	3.92700	0.000412	1.9904	3.92679	0.000412
1	3.92679	0.000002	1.9998	3.92660	0.000002
2	3.92660	0.000000	2.0000	3.92660	0.000000
3	3.92660	0.000000	2.0000		0.000000

Solving by Mathematica software, or Online Equation Solver of Wolfram Alpha the first four non-trivial eigenvalues are:

$$\begin{aligned} \alpha_1 l &= 3.92660246; \alpha_2 l = 7.068583; \\ \alpha_3 l &= 10.2107612; \alpha_4 l = 13.3517688 \end{aligned} \quad (126)$$

The natural frequencies are calculated for $n = 1, 2, 3, 4$ and for $4\beta^4 l^4 = 1, 10, 100, 1000$, and presented in Table 6 together with previous results by Ike (2023), Coşkun (2003), Mutman and Coşkun (2013).

Table 6. Natural frequency parameters for first four modes of transverse vibration of SC thin beams on Winkler foundation.

$K_0 = 4\beta^4 l^4$	λ_1				λ_2			
	Present study	% Difference between present results and GITM results	GITM (Ike 2023)	HPM (Coşkun 2003; Mutman and Coşkun 2013)	Present study	% Difference between present results and GITM results	GITM (Ike 2023)	HPM (Coşkun 2003; Mutman and Coşkun 2013)
1	3.9307	(0%)	3.9307	3.9308	7.0693	(0%)	7.0693	7.0693
10	3.9673	(0%)	3.9673	-	7.0757	(0%)	7.0757	-
100	4.2867	(0%)	4.2869	-	7.1383	(0%)	7.1383	-
1000	5.9314	(0%)	5.9314	-	7.6897	(0%)	7.6897	-

$K_0 = 4\beta^4 l^4$	λ_3				λ_4			
	Present study	% Difference between present results and GITM results	GITM (Ike 2023)	HPM (Coşkun 2003; Mutman and Coşkun 2013)	Present study	% Difference between present results and GITM results	GITM (Ike 2023)	HPM (Coşkun 2003; Mutman and Coşkun 2013)
1	10.2104	(0%)	10.2104	10.2104	13.3519	(0%)	13.3519	13.3519
10	10.2125	(0%)	10.2125	-	13.3529	(0%)	13.3529	-
100	10.2336	(0%)	10.2336	-	13.3623	(0%)	13.3623	-
1000	10.4374	(0%)	10.4374	-	13.4556	(0%)	13.4556	-

4.1.4. Cantilever EBBWF

The natural vibration of EBBWF clamped at the left and $x = 0$ and free at the right end ($x = l$) as shown in Fig. 6 is considered.

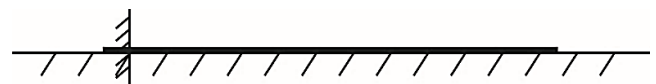


Fig. 6. Natural vibration of EBBWF with clamped end ($x = 0$) and free at $x = l$.

The BCs are:

At the clamped end, $x = 0$,

$$w(x = 0, t) = 0 \quad (127)$$

$$w'(x = 0, t) = 0 \Rightarrow W'(x) = 0 \quad (128)$$

At the free-end, $M(x = l, t) = -EIw''(x = l, t) = 0$

Hence,

$$W''(x = l) = 0$$

$$Q(x = l, t) = -EIw'''(x = l, t) = 0 \quad (129)$$

Hence,

$$W'''(x = l) = 0 \quad (130)$$

Hence,

$$\begin{pmatrix} \left(\frac{\cosh \alpha_n l + \cos \alpha_n l}{2} \right) & \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) \\ \alpha_n \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2} \right) & \left(\frac{\cosh \alpha_n l + \cos \alpha_n l}{2} \right) \end{pmatrix} \begin{pmatrix} W''(0) \\ W'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (131)$$

For nontrivial solutions,

$$\begin{pmatrix} \left(\frac{\cosh \alpha_n l + \cos \alpha_n l}{2} \right) & \left(\frac{\sinh \alpha_n l + \sin \alpha_n l}{2\alpha_n} \right) \\ \alpha_n \left(\frac{\sinh \alpha_n l - \sin \alpha_n l}{2} \right) & \left(\frac{\cosh \alpha_n l + \cos \alpha_n l}{2} \right) \end{pmatrix} = 0 \quad (132)$$

Expanding the determinant gives:

$$\frac{(\cosh \alpha_n l + \cos \alpha_n l)^2}{4} - \frac{(\sinh^2 \alpha_n l - \sin^2 \alpha_n l)}{4} = 0 \quad (133)$$

$$\frac{1}{4}(\cosh^2 \alpha_n l + 2 \cosh \alpha_n l \cos \alpha_n l + \cos^2 \alpha_n l - \sinh^2 \alpha_n l - \sin^2 \alpha_n l) = 0 \quad (134)$$

Using trigonometric identities in Eq. (101), we have:

$$\frac{1}{4}(2 + 2 \cosh \alpha_n l \cos \alpha_n l) = 0 \quad (135)$$

or,

$$\cosh \alpha_n l \cos \alpha_n l = -1 \quad (136)$$

Rewriting Eq. (130) as a standard function, $f(x) = \cosh x \cos x + 1 = 0$

The Newton-Raphson iteration formula for the root is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\cosh x_n \cos x_n + 1}{\sinh x_n \cos x_n - \cosh x_n \sin x_n} \quad (137)$$

Since $f(x = 1) > 0$ and $f(x = 2) < 0$, a real root occurs within the internal $1 < x < 2$.

Let $x_0 = 2$

The Newton-Raphson iteration calculations are presented in Table 7.

Using different initial guesses of the roots yields the successive roots. Second root is obtained as $x = 4.694091$ using the initial guess $x_0 = 4.50$. The third root is obtained as $x = 7.854757$ using the initial guess $x_0 = 7.5$.

The eigenvalues extracted using mathematical software tools, Mathematica and Newton-Raphson's iterative methods for obtaining roots of transcendental equations are:

$$\alpha_1 l = 1.87510; \alpha_2 l = 4.69409; \alpha_3 l = 7.85476; \alpha_4 l = 10.99554; \alpha_n l = \frac{(2n-1)\pi}{2} \quad \text{for } n \geq 5 \quad (138)$$

The dimensionless frequency parameters for the first four modes and for $4\beta^4 l^4 = 1, 10, 100, 1000$, are presented in Table 8.

Table 7. Newton-Raphson iteration for $f(x) = \cosh x \cos x + 1 = 0$.

Iteration n	Approximate root x_n	Function value $f(x_n)$	Derivative value $f'(x_n)$	Error
0	2.000000	-0.565646	-4.930441	-0.565646
1	1.885275	-0.046399	-4.137837	-0.046399
2	1.874062	-0.003926	-4.062095	-0.003926
3	1.875104	0.0000	-4.069150	0.0000
4	1.875104	0.0000	-4.069150	0.0000

4.2. Discussion

This study has explored the Elzaki transform method for the closed form analytical solutions of free harmonic transverse vibration frequencies of prismatic Euler-Bernoulli beams resting on Winkler foundations for classical boundary conditions. The assumption of harmonic response simplified the governing partial differential equation to an ordinary differential equation (ODE) in the spatial variable. The Elzaki transformation of the ODE simplified it to an algebraic equation in the Elzaki transform space. It also considered the initial conditions. The inversion of the Elzaki transformed solution gave

the general solution in terms of the physical (problem) domain variable. The enforcement of boundary conditions gave the characteristic frequency equations for the various BCs cases considered. The expansion of the determinants resulted in transcendental equations whose eigenvalues were used to find the natural frequencies of vibration for each considered BC. The dimensionless eigenfrequencies λ_n for the first five vibratory modes $n = 1, 2, 3, 4, 5$ and for $K_0 = 4\beta^4 l^4 = 1, 10, 100, 1000, 10000$ are presented in Table 1 for EBBWF with simply supported ends. A comparison of present results and previous results shows that the present results are identical with previous results by Ike (2023, 2024b) who used

Generalized Integral Transform Method (GITM), and Kacer et al. (2011). Table 1 further shows that the present

results agree remarkably well with previous results by Rahbar-Ranji and Shahbazzabar (2017) and Zhou (1993).

Table 8. Dimensionless frequency parameters λ_n for the first four transverse vibratory modes of cantilever EBBowF.

$K_0 = 4\beta^4 l^4$	λ_1				λ_2			
	Present study	% Difference between present results and Ike (2023) and Kacer et al. (2011)	Ike (2023)	Kacer et al. (2011)	Present study	% Difference between present results and Ike (2023) and Kacer et al. (2011)	Ike (2023)	Kacer et al. (2011)
1	1.9119	(0%)	1.9119	1.9119	4.6965	(0%)	4.6965	4.6965
10	2.1746	(0%)	2.1746	2.1746	4.7181	(0%)	4.7181	4.7181
100	3.2558	(0%)	3.2558	3.2558	4.9191	(0%)	4.9191	4.9191
1000	5.6407	(0%)	5.6407	5.6407	6.2083	(0%)	6.2083	6.2083

$K_0 = 4\beta^4 l^4$	λ_3				λ_4			
	Present study	% Difference between present results and Ike (2023) and Kacer et al. (2011)	Ike (2023)	Kacer et al. (2011)	Present study	% Difference between present results and Ike (2023) and Kacer et al. (2011)	Ike (2023)	Kacer et al. (2011)
1	7.8553	(0%)	7.8553	7.8553	10.9957	(0%)	10.9957	10.9957
10	7.8599	(0%)	7.8599	7.8599	10.9974	(0%)	10.9974	10.9974
100	7.9058	(0%)	7.9058	7.9058	11.0143	(0%)	11.0143	11.0143
1000	8.3264	(0%)	8.3264	8.3264	11.1790	(0%)	11.1790	11.1790

The eigenfrequency λ_n results for EBBowF with clamped ends are presented in Table 3 for the first four vibratory modes $n = 1, 2, 3, 4$ and for foundation parameter values $K_0 = 4\beta^4 l^4 = 1, 10, 100, 1000, 10,000$. Table 2 shows that the present results for λ_n are identical with previous results that used GITM by Ike (2023), and remarkably identical to previous results by Kacer et al. (2011) and Balkaya et al. (2009). Table 4 also shows the present results for λ_n^2 for EBBowF with both ends clamped for $\rho = A = I = E = 1$ and $4\beta^4 l^4 = 1$. Table 4 confirms that present results for λ_n^2 are identical with previous results that used GITM by Ike (2023) and close to previous results by Mutman and Coşkun (2013), Balkaya et al. (2009) and Chen (2000).

The dimensionless natural frequencies λ_n for EBBowF simply supported at $x = 0$ and clamped at $x = l$ are shown in Table 6 for the first four vibratory modes $n = 1, 2, 3, 4$ and for the foundation parameters $4\beta^4 l^4 = 1, 10, 100, 1000$. Table 6 also compares the present results with previous results by Ike (2023), Coşkun (2003) and Mutman and Coşkun (2013). Table 6 illustrates that the present results by for λ_n are identical to previous results by Ike (2023) and close to previous results by Coşkun (2003) and Mutman and Coşkun (2013) which considered only the case for the value of foundation parameter $K_0 = 4\beta^4 l^4 = 1$.

The results for the dimensionless natural frequency parameters for cantilever EBBowF are presented for the first four vibratory modes $n = 1, 2, 3, 4$ and for foundation parameters $K_0 = 4\beta^4 l^4 = 1, 10, 100, 1000$ in Table 8. Table 8 also displays the previous results by Ike (2023) that used GITM and previous results by Kacer et al. (2011). Table 5 shows that the present results for λ_n are identi-

cal with previous results by Ike (2023) and Kacer et al. (2011). Table 5 shows that the cantilever BC intrinsically yields the lowest natural frequency. This is because the clamped free end does not have intermediate or end supports that constrains transverse deflection, rendering it inherently more flexible when compared to beams with other classical BCs.

The natural transverse vibration frequency results obtained in this work using ETM for all the BCs considered have greatly illustrated the mathematical simplicity, computational efficiency, ease of implementation and broader applicability of the ETM as compared to other previously used methods.

The percentage error between natural vibration frequency results and the results obtained using GITM is nil for all the BC considered. This is because like the GITM, the ETM gives exact analytical solutions to the free vibration problem of EBBowF considered, within the assumptions and theoretical framework of Euler-Bernoulli theory, Winkler foundation model, linear elastic behaviour, small displacement elasticity theory and classical boundary conditions.

When the Winkler foundation parameter tends to zero, the values of natural frequencies approach the values of natural frequencies of corresponding thin beam (without foundations).

Graphical depictions of the results presented in Tables 1, 2, 3, 4 and 5 are shown in Figs. 7, 8, 9 and 10. Fig. 7 is a composite graph that visually illustrates the variations of the first natural frequency λ_1 with the foundation parameter K_0 for the classical boundary conditions considered in this study. Fig. 7 illustrates that for given foundation parameters K_0 , the fundamental natural frequencies

increase in the order: clamped-free (C-free) BC, simply supported (SS) BC, simple-clamped (SC) BC and clamped-clamped (CC) BC. The CC BC gives the highest λ_1 , followed by SC, then SS and the lowest λ_1 is obtained for C-free BCs. Fig. 8 is a composite graph that presents the variation of the first four natural frequencies with foundation parameter K_0 for clamped-clamped EBBowF. Fig. 8 shows that the natural frequencies increase with increase in foundation parameter and increase in vibratory mode.

Fig. 9 presents a composite bar chart showing the variation of the first natural frequencies λ_1 of EBBowF with boundary conditions for foundation parameters $K_0 = 1, 10, 100, 1000$. Fig. 9 shows that the least value of λ_1 is

found for cantilever EBBowF for any given foundation parameter. It also shows that the greatest value of λ_1 is found for clamped-clamped EBBowF for any given foundation parameter. Similarly, Fig. 10 presents a composite bar chart showing the variation of the natural frequencies for the second vibration mode (λ_2) of EBBowF with boundary conditions for values of foundation parameter $K_0 = 1, 10, 100, 1000$. Fig. 10 shows that for the second vibration mode, least λ_2 occurs for cantilever EBBowF for any value of foundation parameter K_0 . Fig. 10 also shows that the greatest value of λ_2 occurs for EBBowF with both ends clamped for any given foundation parameter.

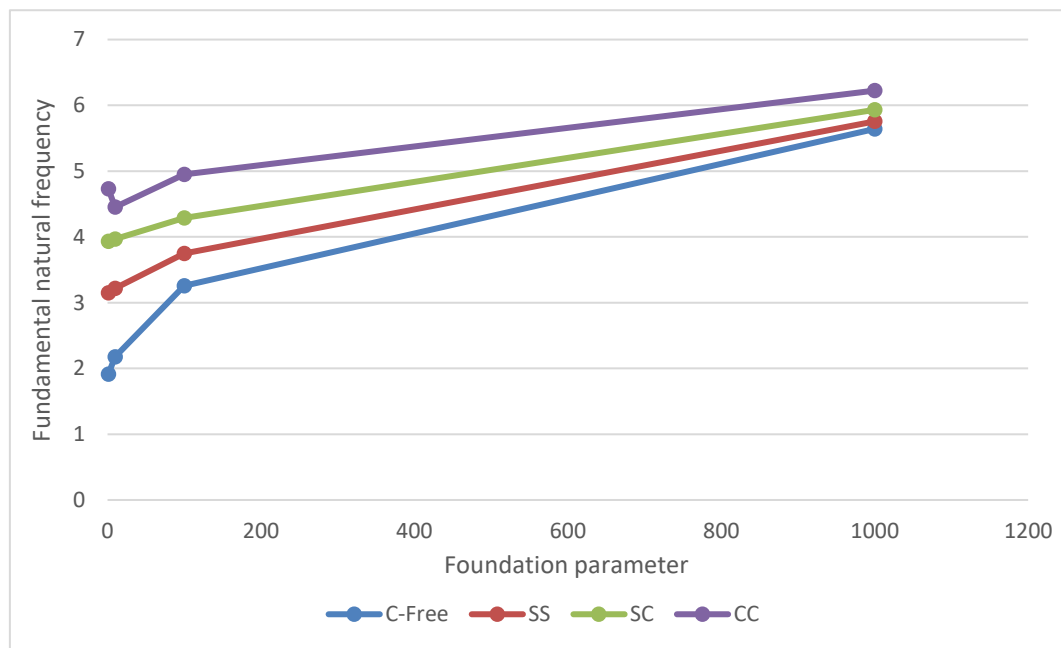


Fig. 7. Graph of variation of λ_1 with K_0 for various support conditions of EBBowF.

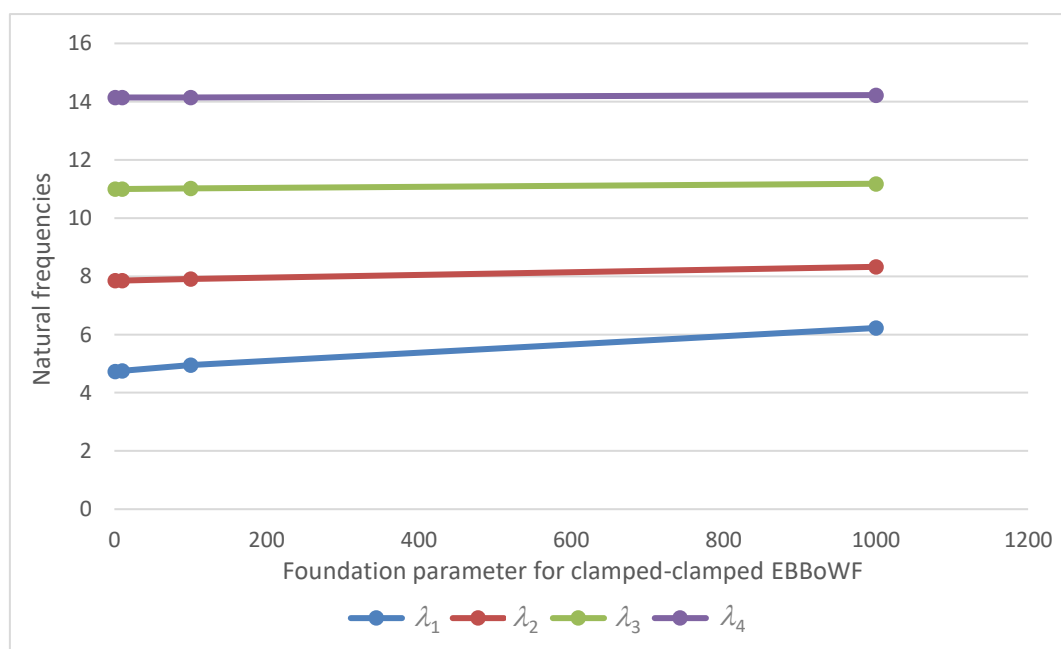


Fig. 8. Variation of first four natural frequencies with K_0 .

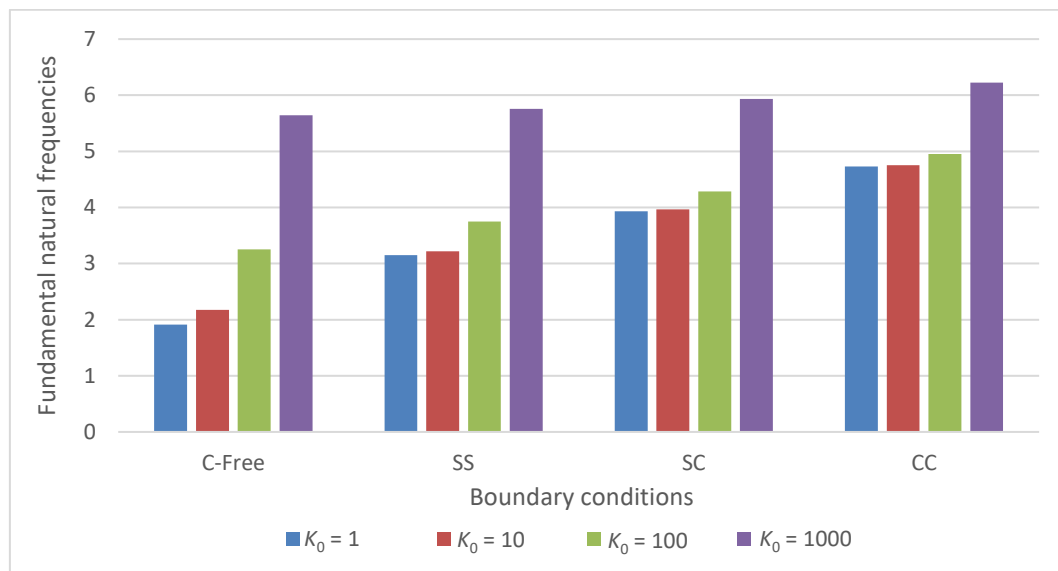


Fig. 9. Composite bar chart showing variation of first natural frequencies λ_1 of EBBowF with boundary conditions for foundation parameters; $K_0 = 1, 10, 100, 1000$.

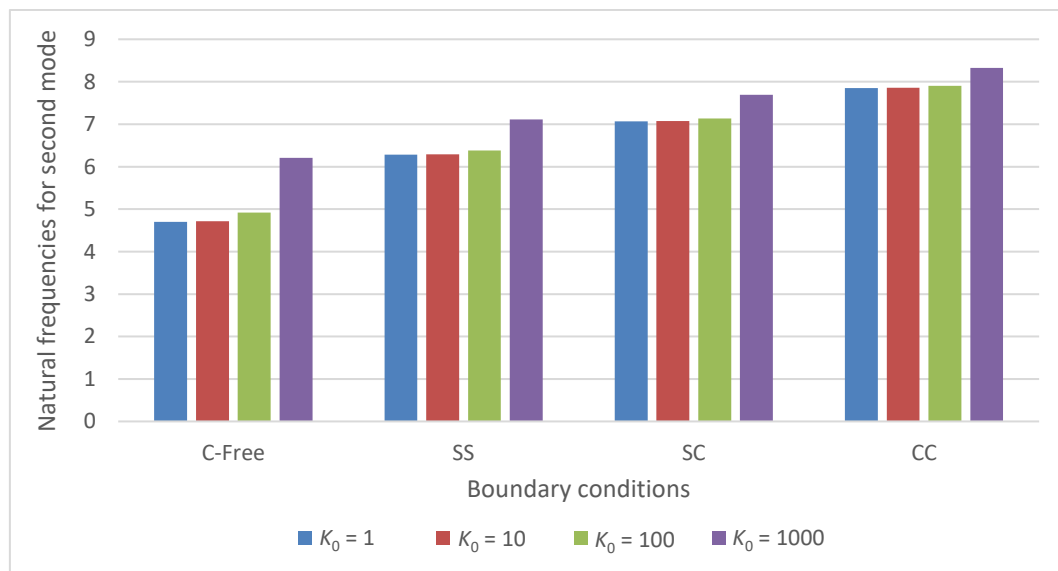


Fig. 10. Variation of natural frequencies for second mode of vibrating EBBowF with different boundary conditions for $K_0 = 1, 10, 100, 1000$.

4.2.1. Discussion on computational efficiency of the ETM and convergence of root extraction iteration schemes

Though the ETM does not require domain mesh refinements, prior selection of basis (trial) functions for BCs satisfaction, spatial discretization or perturbation procedures, numerical methods of root extraction are still required for the transcendental characteristic equations which result when BCs are considered. The ETM offers considerable simplification by converting the governing differential equation into more easily solvable algebraic equations in the frequency domain of Elzaki space. It also natively incorporates initial conditions, removes the necessity for prior general solutions and circumvents the intractable matrix exponentiation requirements of conventional techniques. The simplification is achieved through the core advantages of the integral transform method.

The basic advantage of the ETM and other integral transform methods is that it converts the governing differential equation from a problem of calculus to an algebraic problem which incorporates the initial conditions. Traditional methods of solving differential equations require the determination of the complete general solution by the superposition principle as the sum of the homogeneous and particular solutions, and subsequent applications of initial conditions in order to determine the arbitrary integration constants. The ETM on the other hand directly incorporates the given initial boundary values into the simplified algebraic equation from the initial steps of the formulation. This significantly improves comprehension.

The Elzaki transformation of the governing equation is applied on the entire spatial domain, and hence spatial discretization is not needed. The enforcements of BCs resulted in characteristic equations which are found as

transcendental equations because they contain both trigonometric and hyperbolic functions.

The characteristic eigenfrequency equation for simply supported EBBWF was found as Eq. (85). The characteristic eigenfrequency equation for EBBWF with both ends clamped was found as Eq. (103). The characteristic eigenfrequency equation of EBBWF clamped at the right end and simply supported at the left end was found as Eq. (124).

The characteristic natural frequency equation for cantilever EBBWF was found as Eq. (136). The characteristic natural frequency equations for all the BCs are transcendental equations that are solved for using numerical methods. Numerical methods seek to approximate the roots by starting with an interval or initial guess and repeatedly performing iterations using already established numerical root finding iteration schemes.

The commonly used numerical root extraction methods are (a) Bisection method (Bolzano method), (b) Regula-Falsi method and (c) Newton-Raphson method.

Software applications have been designed for fast implementation of the root extraction of transcendental functions. Some of them are Wolfram Alpha and Wolfram Mathematica, Python, Matlab and Maple. The transcendental equation solvers use Newton-Raphson's numerical iteration algorithms, or other established numerical iteration schemes.

The Newton-Raphson iteration method was used to manually determine the first natural frequency of EBBWF with both ends clamped. The iterations rapidly converged at the third iteration and the error was zero for the root computed to six decimal places. The first five non-trivial roots were determined using Online Wolfram Alpha Equation Solver as given by Eq. (109).

Similarly, the Newton-Raphson iteration method was used to manually compute the first natural frequency of EBBWF with simple support at $x = 0$ and clamped support at $x = l$. The Newton-Raphson's iteration calculations are presented in Table 5 for an initial guess at $x_0 = 3.927$. Table 5 shows a rapid convergence after two iterations with an error of zero and eigenvalue of $x = 3.92660$ to five decimal places. The first four natural frequencies were determined for the SC boundary conditions using Online Wolfram Alpha Equation Solver as Eq. (126).

The Newton-Raphson iteration solution of the characteristic frequency equation is presented in Table 7 for an initial guess of $x_0 = 2.0$. Table 7 relied on Eq. (137) to compute successive approximations of the root of the characteristic frequency equation. Table 7 shows that the root converged at the third iteration with zero error, illustrating the rapidly convergent nature of the Newton-Raphson iteration. The first four non-trivial eigenvalues were found using Online Wolfram Alpha Equation solver and other mathematical software tools as Eq. (138). The dimensionless forms of the natural frequencies for each considered BC were then obtained from α_n/l by using Eq. (115) and inputting the various values of the foundation parameters.

The application of ETM to EBBWF can be complicated by foundation discontinuities, variable Winkler foundation moduli and complex inverse transformations, all of which limit its scope and effectiveness in

the absence of supplementary numerical techniques. ETM inherently assume smooth continuous functions across the beam's longitudinal axis. Some practical beam foundation problems involve discontinuous loads, variable beam cross-sections and non-classical BCs. Discontinuous loads, variable cross-sections and non-classical BCs disrupt the transformation process and introduce piecewise functions or Dirac delta functions, thereby complicating the transformation to an algebraic problem. Also, when the Winkler foundation modulus varies along the longitudinal (x) axis, thus making K become dependent on x , the governing equation would contain variable parameters. It would be very difficult to apply the ETM to such variable parameter differential equations without converting them to complex differential forms, thus defeating the initial objective of a swift transform solution.

5. Conclusions

This work has explored the ETM for the exact analytical solutions of free harmonic transverse vibrations of EBBWF for the following BCs: simply supported at both ends, clamped at both ends, simply supported at left end and clamped at right end, and clamped at left end and free at the right. In conclusion:

- The assumption of harmonic response decouples the PDE to a fourth order ODE in terms of the space variable.
- Application of the ETM to the ODE gives an integral equation which further simplifies to an algebraic equation in the Elzaki transform parameter. Solving and inverting the solution in the transformed space gives the general solution.
- Applying the BCs for the four considered cases gave the characteristic frequency equation as a transcendental equation whose roots are extracted using mathematical software tools such as Wolfram Mathematica, Mathcad or iterative methods such as the Newton-Raphson iteration.
- The present eigenfrequency solutions for simply supported EBBWF are exact within the assumptions and theoretical framework of Euler-Bernoulli theory, Winkler foundation model, linear elastic behaviour, small displacement elasticity theory and classical boundary conditions (for all five modes and foundation parameters).
- The present eigenfrequency solutions are identical with previous solutions obtained using Generalized Integral Transform Method (GITM) and Stodola-Vianello iteration method (SVIM), and are close to previous solutions based on DTM and Rayleigh-Ritz method using Legendre polynomials as basis functions.
- For all the four vibration modes, and foundation parameters considered, the present eigenfrequency (natural frequency) solutions for the other classical BCs are exact within the assumptions and theoretical framework of Euler-Bernoulli theory, Winkler foundation model, linear elastic behaviour, small displacement elasticity theory and classical boundary conditions.

tions and are identical with previous solutions obtained using GITM and close to previous solutions using DTM.

- The natural frequencies increase as the foundation parameters $K_0 = 4\beta^4 l^4$ increase for all the vibration modes and for all the classical boundary conditions considered. This is because the elastic foundation increases the dynamic stiffness of the beam on elastic foundation system, thus raising the natural frequency.
- The key advantages of ETM include the following:
 - It does not require domain discretization.
 - The BC that produces the lowest natural frequencies is the cantilever BC. This is because it lacks intermediate or boundary support, making it more flexible than other BCs.
 - The BC that produces the highest natural frequency is the clamped-clamped BC. This is because the clamped-clamped ends restrict transverse deflection, making the structure inherently more rigid.
 - The effects of the elastic foundation on the first few natural frequencies are to increase the natural frequencies of vibration.
- The ETM has potential future extensions to other more complex applications, such as non-uniform beam, Timoshenko beam theory, non-linear foundations, damping effects and thick beams.

5.1. Limitations of the ETM

The ETM is a good mathematical tool for solving linear ordinary and partial differential equations. However, its main limitations include the difficulties in extending to nonlinear partial differential equations without hybrid modifications, restricted use to complex initial conditions, lack of widespread software support unlike the established Laplace or Fourier transforms.

5.2. Recommendations for future research

In order to overcome some of the limitations of ETM, further research could explore the use of recently developed generalized and modified variants of ETM, such as Fractional Elzaki transforms and Elzaki decomposition methods EDMs. Then future research could explore the applications to studies of nonhomogeneous EBBowF, discontinuous EBBowF, and EBBowF with nonclassical BCs.

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Appendix A. Elzaki transform table

$$T(x^n) = n! u^{n+2}$$

$$T e^{ax} = \frac{u^2}{1-au}$$

$$T \sin(ax) = \frac{au^3}{1+a^2u^2}$$

$$T \cos(ax) = \frac{au^2}{1+a^2u^2}$$

$$T \sinh(ax) = \frac{au^3}{1-a^2u^2}$$

$$T \cosh(ax) = \frac{au^2}{1-a^2u^2}$$

Hence,

$$T^{-1}u^n = \frac{x^{n-2}}{(n-2)!}; \quad n = 2, 3, 4, \dots$$

$$T^{-1}\left(\frac{u^2}{1-au}\right) = a^{ax}$$

$$T^{-1}\frac{u^3}{1+a^2u^2} = \frac{1}{a} \sin(ax)$$

$$T^{-1}\frac{u^2}{1+a^2u^2} = \frac{1}{a} \cos(ax)$$

$$T^{-1}\frac{u^3}{1-a^2u^2} = \frac{1}{a} \sinh(ax)$$

$$T^{-1}\frac{u^2}{1-a^2u^2} = \frac{1}{a} \cosh(ax)$$

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Data Availability

The datasets generated and/or analyzed during the current study are not publicly available but are available from the corresponding author upon reasonable request.

AI Assistance

No AI-based tools were used in the preparation of this manuscript.

Author Contributions

All authors made substantial contributions to the conception and design of the study, acquisition of data, analysis and interpretation of data; drafted or critically revised the manuscript for important intellectual content; and approved the final version to be published.

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