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Research on relation between natural frequency and axial stress of round bar with intermediate-supported ends

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ABSTRACT

In order to make a method be useful to measure an axial stress of a member by a natural frequency, we investigated a relation between a natural frequency and an axial stress of a round bar with intermediate-supported ends, the boundary condition of which was one between a fix-supported end and a simply-supported end. To define an intermediate-supported end condition, we adopted a parameter, a ratio of a moment of a force to a deflection angle at the end. It was shown theoretically that the parameter of an intermediate-supported end could be evaluated by one at a support on a continuous beam consisted of 3 spans. The 3-spanned beam has same vibration characteristics of a beam with intermediate-supported ends. We manufactured a test device of a 3-spanned beam by which we could simulate a vibration under various intermediate-supported end conditions. The theoretical relation and experimental results between a natural frequency and an axial stress agreed for the most part.

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1. Introduction

From a view point of energy-saving, economic merits and environmental friendliness, development of health assessment methods for an existing structure or a machine is one of urgent problems. Fair evaluation of their integrity enables us to use them extending over an established design life.

Strain gauges have been used to measure strains in a structure or a machine. The method is simple and reliable. New kinds of strain gauges like a semiconductor or a piezoelectric element have been developed. Fiber Bragg Grating sensor was employed for health monitoring for a structure or a composite material (Moreira et al., 2012). In order to apply the method to a structure or a machine, we need many gauges. The measurement by strain gauges takes a lot of labour and accompanies troublesome works, and is unsuitable to measure working stress over a long period.

On the one hand, methods to measure strain at one point have been developed, on the other hand, methods to estimate strain distribution in an area have been developed. They are a digital image correlation method (Leplay et al., 2011) or an acoustic-elastic method (Kudryavtsev, 2008). These methods need to compare a picture or a property before and after the deformation of a structure or a specimen. The procedure is still complicated and is not easily employed on site.

Bars or rods play an important role in current structures as shown in Fig. 1. Being fabricated into a structure, they are expected to cooperate to suspend heavy burden. In order to arrange stresses in bars or rods within allowable level during construction or in operation, it is necessary to be able to measure their axial stress easily, reliably and promptly.

We have investigated a relation between a natural frequency and an axial stress of a round bar with fix-supported ends or with simply-supported ends. Theoretical relations and experimental results agreed (Yoshida et al., 2010). The agreement enabled us to measure an axial stress of a bar under these ends by a natural frequency. But, when we measured an axial stress of bars in an experimental truss device (Yoshida et al., 2013), experimental results agreed neither with a theoretical relation under fixed ends nor with one under simply-supported ends. The truss bar vibrated under the ends between

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simply-supported ends and fix-supported ends. This is not only true of the experimental truss bar, but also of any member in existing structures or machines.

Fig. 1. Current structure.

In this research, a relation between a natural frequency and an axial stress was investigated for the beam with the ends between fixed supported ends and simply-supported ends. We refer such the end as an intermediate-supported end. To characterize the end condition, we adopted a parameter, a ratio of a moment of a force to a deflection angle at the end. To assess an intermediate-supported end condition, we employed a continuous beam consisted of 3 spans. Employing the 3-spanned beam, we compared theoretical relations and experimental results between a natural frequency and an axial stress under various intermediate-supported end conditions.

2. Theory and Numerical Calculation

2.1. Single beam with intermediate-supported ends

To explain a boundary condition of an intermediate-supported end, we illustrated a beam with the ends in Fig. 2 (Jinbo and Furukawa, 1971). The beam is supported by rigid sharp edges and is able to incline at the end. The beam extends into a resilient material. If the material were hard, the vibration of the beam would be one with fix-supported ends. If the material were soft, the vibration would be one with simply-supported ends. An intermediate hardness of a resilient material makes the beam vibrate with ends between simply-supported ends and fix-supported ends.

Boundary conditions at the ends are given by the following expressions.

$$(w)_{x=0} = 0\,, \quad (w)_{x=L} = 0\,, \tag{1}$$

$$-EI\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = k\left(\frac{\partial w}{\partial x}\right)_{x=0}\,,$$

$$-EI\left(\frac{\partial^2 w}{\partial x^2}\right)_{x=L} = -k\left(\frac{\partial w}{\partial x}\right)_{x=L}\,. \tag{2}$$

Here, w is a deflection of a beam and EI is a flexural rigidity. k is the ratio between a moment of a force and a deflection angle at the end. We refer k as a resilient parameter and a beam with intermediate-supported ends as an intermediate beam.

If k were zero, the moment of a force at the end, the left hand side of the Eqs. (2), should be zero. Then the boundary condition of the end corresponds to that of a simply-supported end. If k were quite large and there occurred a finite quantity of moment of a force at the end, the deflection angle of the beam at the end should be small. The boundary condition corresponds to that of a fix-supported end. The value of k between zero and quite a large value defines a boundary condition of an intermediate-supported end.

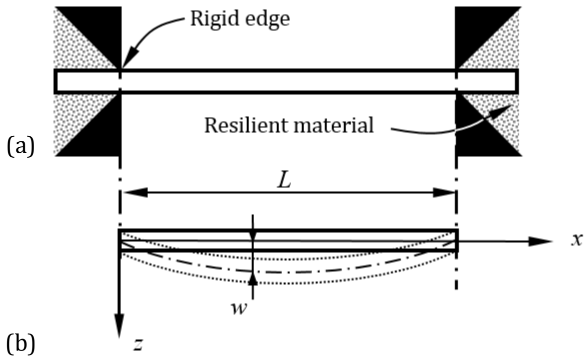

Fig. 2. Intermediate-supported ends: (a) Concrete instance; (b) Coordinate and deflection.

Following a conventional procedure to obtain a natural frequency of a beam, the frequency equation of the beam with intermediate-supported ends is given by the next formulae in the form of a determinant. The conventional procedure to obtain a natural frequency is explained in detail elsewhere (Timoshenko, 1954). Here ω is an angular frequency, ρ is a density and A is a sectional area.

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \lambda & \Lambda & -\lambda & \Lambda \\ \cos(\chi) & \sin(\chi) & \cosh(\chi) & \sinh(\chi) \\ \lambda\cos(\chi) + \Lambda\sin(\chi) & \lambda\sin(\chi) - \Lambda\cos(\chi) & -\lambda\cosh(\chi) - \Lambda\sinh(\chi) & -\lambda\sinh(\chi) - \Lambda\cosh(\chi) \end{vmatrix} = 0\,, \tag{3}$$

where $\Lambda = k/EI$, $\chi = \lambda \cdot L$, $\lambda^2 = \omega/\eta$ and $\eta^2 = EI/\rho A$.

2.2. Three-spanned beam

To illustrate an intermediate-supported end, we assumed a resilient material. But, it is difficult to find out such material and achieve the end practically. We took up a continuous beam shown in Fig. 3(a). The beam is fixed at both ends and is simply-supported at two inner supports. The beam consists of 3 members: side members and a center member. A natural frequency of the beam varies depending on the length of the side members. We refer the beam as a 3-spanned beam. We show later that the 3-spanned beam has same vibration characteristics with an intermediate beam.

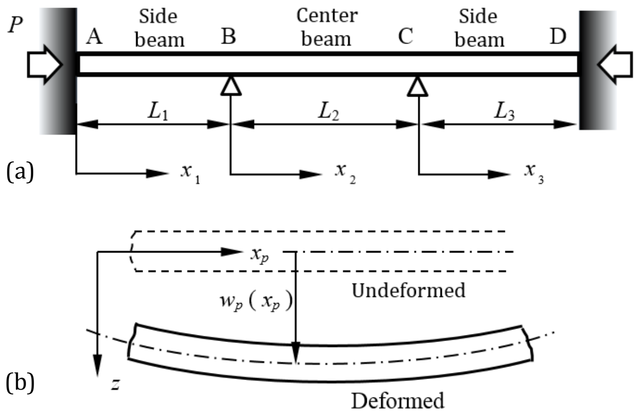

Fig. 3. Three-spanned beam with axial load: (a) Coordinate; (b) Deflection.

In order to investigate the relation between a natural frequency and an axial stress of the beam, we analysed the beam applying an axial load. A differential equation of a motion of each member of a 3-spanned beam applied with an axial load, P is given by the following equation for a transverse vibration of a uniform section and density.

$$\frac{\partial^2 w_p}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 w_p}{\partial x_p^4} + \frac{P}{\rho A}\frac{\partial^2 w_p}{\partial x_p^2} = 0\,. \tag{4}$$

A subscript, p (p=1,2,3) was employed to distinguish each member of the beam. Here, x_p and t are coordinate and time variables respectively. w_p is a deflection of each beam shown in Fig. 3(b). A solution of Eq. (4) was assumed to be given by the following form as the product of a modal shape function and time variation.

$$w_p(x_p,t) = W_p(x_p)\cdot e^{j\omega t}\,, \tag{5}$$

where j is an imaginary unit.

The mode shape may be written by the next form,

$$W_p(x_p) = C_{p1}\cdot\cos(\lambda_1 x_p) + C_{p2}\cdot\sin(\lambda_1 x_p) + C_{p3}\cdot\cosh(\lambda_2 x_p) + C_{p4}\cdot\sinh(\lambda_2 x_p)\,, \tag{6}$$

where,

$$\lambda_1 = \sqrt{\frac{\sqrt{\pi^4\cdot\beta^2+4\omega^2/\eta^2}+\pi^2\cdot\beta}{2}}\,,$$

$$\lambda_2 = \sqrt{\frac{\sqrt{\pi^4\cdot\beta^2+4\omega^2/\eta^2}-\pi^2\cdot\beta}{2}} \quad \text{and} \quad \beta = \frac{P}{EI\pi^2}\,. \tag{7}$$

Twelve boundary conditions are provided for a 3-spanned beam. For example, the deflection at the left ends of each member is zero, which is expressed by Eq. (8a). The angle of a deflection or the moment of a force on either side of inner supports are to be same. The conditions are given by Eq. (8b).

$$W_p(0) = 0\,, \tag{8a}$$

$$W_p'(L_p) = W_{p+1}'(0)\,,\; W_p''(L_p) = W_{p+1}''(0)\,,\; p = 1{,}2 \tag{8b}$$

Applying these boundary conditions to Eq. (6), the following simultaneous equation, unknown variables of which are C_{pk} (k =1,2,3,4), is obtained. Here $\{C_{pk}\}$ stands for a row vector with 12 elements and $[\mathbf{K}]$ is a 12x12 matrix of coefficients of the simultaneous equation.

$$[\mathbf{K}]\{C_{pk}\} = \{\mathbf{0}\}\,. \tag{9}$$

The eigenvalue problem, Eq. (9), has a nontrivial solution only if the determinant of the matrix $\mathbf{K}$ vanishes.

$$|\mathbf{K}| = 0\,. \tag{10}$$

Eq. (10) gives the frequency equation of a 3-spanned beam.

A resilient parameter at the support, B in Fig. 3(a) is given by the following expression according to the definition.

$$k = \frac{M}{\theta} = -\frac{EI\cdot W_2''(0)}{W_2'(0)} = EI\cdot\frac{C_{21}\cdot(\lambda_1^2+\lambda_2^2)}{\lambda_1\cdot C_{22}+\lambda_2\cdot C_{24}}\,. \tag{11}$$

2.3. Numerical result

We used, in common, an 8 mm diameter round bar and material properties of a steel, such as E=206 GPa and ρ=7.89 g/cm^3 through this research. A simple analysis teaches that a round bar made out of a steel with 8 mm diameter and 200 mm span length vibrates with 401.3 Hz under simply-supported ends and with 909.7 Hz under fix-supported ends as shown in Table 1.

Table 1. Parameters of various ends.

Supported ends	f [Hz]	k [N·mm]	Ls [mm]
Simply-supported ends	401.3	$<10^6$	250
Intermediate-supported ends	↕	↕	↕
Fix-supported ends	909.7	$>10^{12}$	0

2.3.1. Relation between resilient parameter and natural frequency of intermediate beam

Employing Eq. (3), we calculated a relation between a resilient parameter and a natural frequency of the beam with intermediate-supported ends shown in Fig. 2. The beam taken up for the calculation is a round bar with steel properties, 8 mm diameter and 200 mm span length. Fig. 4(a) shows the result. A horizontal axis is a resilient parameter, k by a logarithmic scale and a vertical axis is a natural frequency.

When k becomes smaller than 10^6, the beam vibrates with the natural frequency, 401.3 Hz. The figure is same with that of the natural frequency under simply-supported ends in Table 1. When k becomes larger than 10^{12}, the beam vibrates with the natural frequency, 909.6 Hz. The figure is almost same with the natural frequency under fix-supported ends in Table 1. When the beam is supported by the ends with a resilient parameter between 10^6 and 10^{12}, the beam vibrates with a natural frequency between 401.3 to 909.6 Hz. The vibration is neither one under simply-supported ends nor one under fix-supported ends, but one under intermediate-supported ends. We can define an intermediate-supported end condition by its corresponding resilient parameter as shown in Table 1. A resilient parameter defines a condition of an end from a simply-supported end to a fix-supported end.

2.3.2. Relation between side span length and natural frequency of 3-spanned beam

Employing Eq. (10), we calculated a relation between a length of the side members and the first mode natural frequency of the 3-spanned beam with no axial stress and showed it by the solid line in Fig. 4(b). The beam taken up for the calculation is a round bar with steel properties, 8 mm diameter and, a length of a center member was set to be 200 mm. A horizontal axis is a length of the side members and a vertical axis is a natural frequency.

When the length of the side members approaches to 250 mm, a natural frequency of the beam approaches to 401.3 Hz. The figure is same with that of the natural frequency under simply-supported ends in Table 1. The length approaches to zero, the natural frequency approaches to 909.7 Hz. The figure is same with the above natural frequency under fix-supported ends. When the length of the side members is one between 0 and 250 mm, the beam vibrates with its corresponding natural frequency between 401.3 and 909.7 Hz as shown in Table 1. The 3-spanned beam has the same natural frequency range of an intermediate beam.

2.3.3. Equivalence of supported end on intermediate beam and 3-spanned beam

The natural frequency range as well as the vertical axis length of Figs. 4(a, b) were adjusted to be same. Fig. 4(a) refers to an intermediate beam and Fig. 4(b) to a 3-spanned beam. The material and the diameter of the two beams are same. The span length of the intermediate beam in Fig. 4(a) and the length of the center member of the 3-spanned beam in Fig. 4(b) are same 200 mm. Following the arrows extending over Fig. 4(a) to Fig. 4(b), we can correlate a resilient parameter to the side member length as like k_x to L_{sx} through the common natural frequency f_x. The reverse is also true. The arrows are connecting the vibration characteristics of the two beams.

In the calculation of the 3-spanned beam, we evaluated a resilient parameter, k at the support B in Fig. 4(b) defined by Eq. (11). Onto Fig. 4(a), we put the obtained values by square marks. Resilient parameters at the support on a 3-spanned beam agreed with those at the end on an intermediate beam theoretically. Equivalence of an intermediate-supported end on two beams was demonstrated.

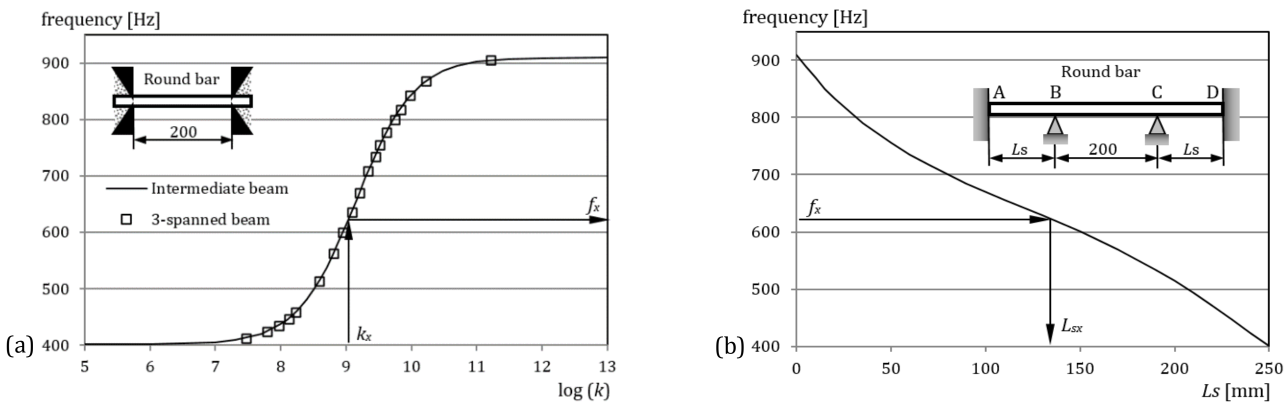

Fig. 4. Relation between resilient parameter, length of side members and natural frequency: (a) Intermediate beam; (b) 3-spanned beam.

3. Experiment

3.1. Experimental device

We manufactured a test device of a 3-spanned beam shown in Fig. 5. A round bar with 8 mm diameter made out of a steel was used as a specimen. Both ends of the specimen were fixed. Two inner supports, the condition of which was that of a simply-supported end, were placed between the fixed ends. The supports make the specimen be partitioned into side members and a center member. The length of the center member between the inner supports was set to be 200 mm. The length of both side members, *Ls* is alterable to be 25, 75, 125, 175 and 252.2 mm. Frequencies of the specimen vibration vary according to the length, *L*s. An axial stress can be loaded to the specimen by rotating a nut at its end. After an axial stress was loaded, setups for fix-supported ends and inner supports were made by clamping their bolts.

Fig. 5 also shows the measurement system. The system consists of a microphone, an amplifier, a filter, a fast fourie transformation (FFT) analyser and the 3-spanned beam device. In the experiment, striking the center member by a wooden bar, we generated a sound. Measuring the sound by a microphone, we analysed the sound through the FFT analyser and obtained natural frequencies of the 3-spanned beam.

3.2. Relation between axial stress and natural frequency under intermediate-supported ends

Theoretical relations and experimental results between an axial stress and a natural frequency under intermediate-supported end conditions were compared. The results are shown in Fig. 6. Horizontal axis is an axial stress. Vertical axis is a natural frequency. Lines show the theoretical relations obtained by Eq. (10). The experiment was conducted by the device shown in Fig. 5. Experimental data were shown by symbols.

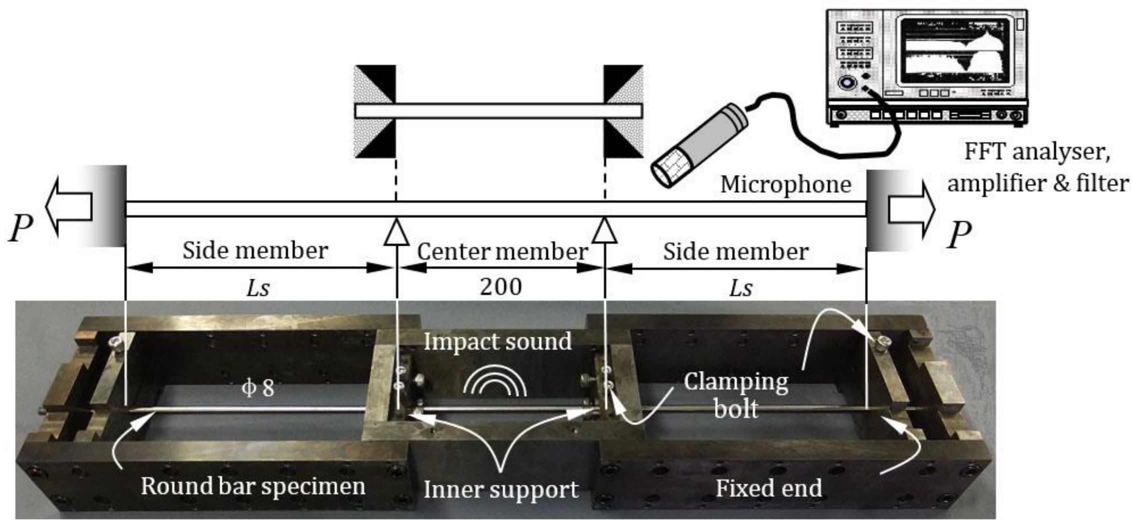

Fig. 5. Experimental device of 3-spanned beam with measurement system.

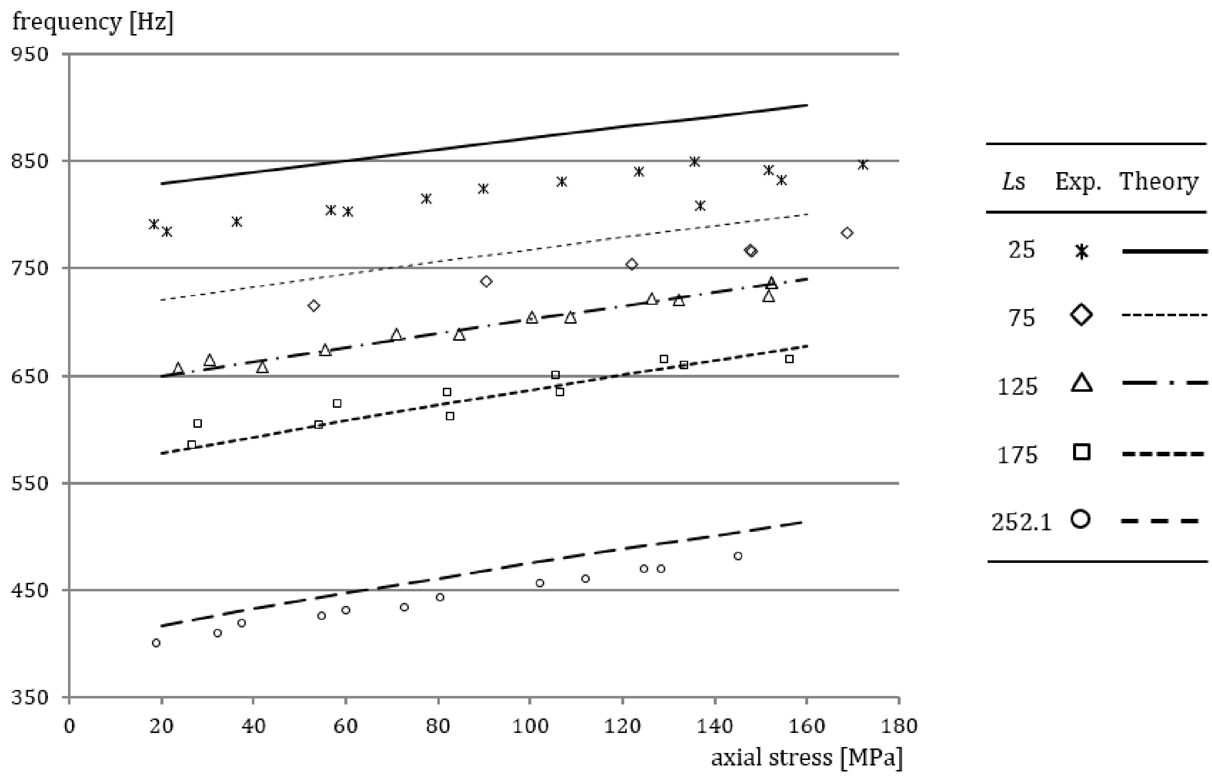

Fig. 6. Relation between axial stress and natural frequency for various side lengths.

The theoretical relations and experimental data agreed except one for Ls=20. Higher machining accuracy is needed for the experiment with shorter Ls. The accordance between the theory and the experiment enables us to estimate an axial stress by a natural frequency under intermediate-supported ends.

Ls is related to k as explained in the previous section. A practical application of the method to measure an axial stress of a beam with intermediate-supported ends by a natural frequency demands to evaluate a resilient parameter value at the end of the beam experimentally. We are now trying to evaluate a resilient parameter at the simply-supported end of the 3-spanned beam measuring deflections and strains. We left the development for the evaluation as future work.

4. Conclusions

A beam used in a structure or a machine is supported neither by a simply-supported end nor by a fix-supported end, but one between them. The support condition has a great influence on a natural frequency of the beam. We have been developing a method to measure an axial stress of a beam by its natural frequency. It is necessary to make it clear the relation between a natural frequency and a support-end condition. To define a support-end condition, we adopted a parameter, a ratio between a moment of a force and a deflection angle at an end.

- We developed a theory of a continuous beam with three spans and a beam with intermediate-supported ends.
- It was shown theoretically that an intermediate-supported end condition could be simulated by one at a support on a three-spanned beam.
- Applying an axial stress to a three-spanned beam, a relation between natural frequencies and axial stresses was investigated under various intermediate-supported end conditions experimentally and theoretically. They agreed for the most part. It implies that we can measure an axial stress by a natural frequency under intermediated-supported ends.

We left as future work an experimental procedure to measure a parameter value which would define an intermediate-supported end condition.

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