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Viscoelastic rod using the generalized finite difference method

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ABSTRACT

The finite difference method is quite extensively used to obtain the approximate solutions of many equations of mathematical physics. In this study, the precise algorithm in the time domain is combined with the generalized finite difference method to solve dynamic viscoelasticity problems. The numerical results obtained are satisfactory, and they are presented together with finite difference and finite element solutions.

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1. Introduction

The classical finite difference method is used to solve a large variety of engineering problems. However, when the region under consideration gets complicated, the classical method becomes difficult to apply since it involves the use of irregular grids. In many structural mechanics problems, for example large deformations, and propagation of crack, the domain's geometry changes continuously, which can make analyzing problems challenging and expensive. Meshless methods provide an alternative technique to explore these issues and eliminate mesh-related problems. The generalized finite difference method (GFDM), which is also called meshless finite difference method (Jaworska and Milewski 2016; Fu et al. 2020), on the other hand, can readily use irregular grids or clouds of points with high efficiency. This capability of the method extends its applicability to a wider range of problems. The main ideas behind this meshless method were given in the seventies. Nevertheless, fully arbitrary meshes were first used by Jensen (1972). Perrone and Kao (1975) suggested some modifications to improve the behavior of the coefficient matrix. Liszka and Orkisz (1980) introduced the use of the finite difference method at arbitrary irregular grids.

Benito et al. (2001) conducted a study to examine the impact of the primary parameters involved. In this meshless method Taylor series expansion of functions are used to compute approximations to derivatives at some specified points. These approximations are then used to solve differential equations in an approximate manner. Of course, the precision of the solution obtained can readily be improved by adding more and more terms in the Taylor series expansion and/or by adding more points into the domain.

The formulation given in the next section is for one-dimensional problems. Three terms in the Taylor series expansion are retained in the derivations. Of course, when only two terms are retained, the solution obtained would be less accurate. Generalized finite differences can easily be extended to higher dimensional problems. It is to be noted that generalized finite differences are used to integrate the governing differential equation of the problem with respect to space variables. The resulting second-order system is first converted to an equivalent system of first-order ordinary differential equations. Then the resulting system is solved using an implicit Runge-Kutta method (Radau) of order 5 with step size control (Hairer and Wanner 1991).

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2. Generalized Finite Difference Method

For a sufficiently differentiable function $f(x)$, defined in a region $0 \le x \le L$, the Taylor series expansion around a point $P(x_i)$ is

$$f(x) = f_i + (x - x_i)\frac{df_i}{dx} + \frac{1}{2}(x - x_i)^2\frac{d^2 f_i}{dx^2} + \frac{1}{6}(x - x_i)^3\frac{d^3 f_i}{dx^3} + O(x - x_i)^4 \quad (1)$$

where $f_i = f(x_i)$. One can define the error the above Taylor series expansion as $e(x) = f_i - f(x)$ and the error computed at point $Q(x_j)$ is e_j. Let $\omega(x)$ be a weighting function with compact support, then one can define norm *B*

$$B = \sum_{\substack{j=1 \\ i \ne j}}^{N}\left(e(x_j)\omega_j\right)^2 \quad (2)$$

substituting for $e(x_j)$ gives,

$$B = \sum_{\substack{j=1 \\ i \ne j}}^{N}\left[\left[f_i - f_j + (x_j - x_i)\frac{df_i}{dx} + \frac{1}{2}(x_j - x_i)^2\frac{d^2 f_i}{dx^2} + \frac{1}{6}(x_j - x_i)^3\frac{d^3 f_i}{dx^3}\right]\omega_j\right]^2 \quad (3)$$

where *N* is the total number of points in the region. Approximations to df_i/dx, $d^2 f_i/dx^2$, and $d^3 f_i/dx^3$ can be computed by minimizing norm *B*,

$$\frac{\partial B}{\partial\{Df\}} = 0 \quad (4)$$

$$\{Df\}^T = \left\{\frac{df}{dx}, \frac{d^2 f}{dx^2}, \frac{d^3 f}{dx^3}\right\} \quad (5)$$

which yields a set of three equations with three unknowns for each point $P(x_i)$. These equations are

$$\begin{bmatrix} \sum\omega_j^2 h_j^2 & \sum\omega_j^2\frac{h_j^3}{2} & \sum\omega_j^2\frac{h_j^4}{6} \\ \sum\omega_j^2\frac{h_j^3}{2} & \sum\omega_j^2\frac{h_j^4}{4} & \sum\omega_j^2\frac{h_j^5}{12} \\ \sum\omega_j^2\frac{h_j^4}{6} & \sum\omega_j^2\frac{h_j^5}{12} & \sum\omega_j^2\frac{h_j^6}{36} \end{bmatrix}\begin{Bmatrix} df_i/dx \\ d^2 f_i/dx^2 \\ d^3 f_i/dx^3 \end{Bmatrix} = \begin{pmatrix} -f_i\sum\omega_j^2 h_j + \sum f_j\omega_j^2 h_j \\ -f_i\sum\omega_j^2\frac{h_j^2}{2} + \sum f_j\omega_j^2\frac{h_j^2}{2} \\ -f_i\sum\omega_j^2\frac{h_j^3}{6} + \sum f_j\omega_j^2\frac{h_j^3}{6} \end{pmatrix} \quad (6)$$

The above system of linear equations might be presented in matrix form:

$$A_P D_P = b_P \quad (7)$$

where A_P is a 3×3 matrix and b_P is a 3×1 vector. For computational purposes the right hand side in Eq. (6) is expressed in the form

$$b_P = B_P\tilde{f} \quad (8)$$

where *B* is a $3\times(N+1)$ matrix

$$B_P = \begin{bmatrix} -\sum\omega_j^2 h_j & \omega_1^2 h_1 & \omega_2^2 h_2 & \cdots & \omega_{N-1}^2 h_{N-1} & \omega_N^2 h_N \\ -\sum\frac{1}{2}\omega_j^2 h_j^2 & \frac{1}{2}\omega_1^2 h_1^2 & \frac{1}{2}\omega_2^2 h_2^2 & \cdots & \frac{1}{2}\omega_{N-1}^2 h_{N-1}^2 & \frac{1}{2}\omega_N^2 h_N^2 \\ -\sum\frac{1}{6}\omega_j^2 h_j^3 & \frac{1}{6}\omega_1^2 h_1^3 & \frac{1}{6}\omega_2^2 h_2^3 & \cdots & \frac{1}{6}\omega_{N-1}^2 h_{N-1}^3 & \frac{1}{6}\omega_N^2 h_N^3 \end{bmatrix} \quad (9)$$

and $\tilde{f}$ is an $(N+1)\times 1$ vector

$$\tilde{f}^T = \begin{pmatrix} f_i & f_1 & f_2 & \cdots & f_{N-1} & f_N \end{pmatrix} \tag{10}$$

Eq. (6) can be inverted to compute derivatives df_i/dx , $d^2 f_i/dx^2$, and $d^3 f_i/dx^3$ at point $P(x_i)$ in terms of nodal values of $f(x)$, that is, in terms of vector $\tilde{f}$ as follows

$$\begin{Bmatrix} df_i/dx \\ d^2 f_i/dx^2 \\ d^3 f_i/dx^3 \end{Bmatrix} = C_P \tilde{f} \tag{11}$$

where $C_P = A_P^{-1} B_P$.

3. Governing Equations

Governing equations of a dynamic viscoelastic problem are

$$\sigma_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial x^2} \quad x \in R \tag{12}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{13}$$

where σ_{ij} and ε_{ij} represent stress and strain tensors, respectively, u_i is the displacement vector, ρ is the density, F_i is the body force, and R is the domain of definition of the problem. In these expressions, *i* and *j* range from one to dimension of the problem (that is to 1 for one-dimensional problems, to 2 for two-dimensional problems). The boundary conditions are specified along the boundaries of R in the form

$$u_i = \tilde{u}_i \quad x \in \Gamma_u \tag{14}$$

$$\sigma_{ij} n_j = \tilde{p}_i \quad x \in \Gamma_\sigma \tag{15}$$

where n_j is the unit outward vector, $\tilde{p}_i$ refers to the prescribed traction vector on the boundary, $\Gamma = \Gamma_u + \Gamma_\sigma$ denotes the boundary of the region R. The initial conditions are

$$u_i = \tilde{u}_i^0(x) \; at \; t = 0 \tag{16}$$

$$\frac{\partial u_i}{\partial t} = \tilde{\upsilon}_i^0(x) \; at \; t = 0 \tag{17}$$

here $\tilde{u}_i^0(x)$ and $\tilde{\upsilon}_i^0(x)$ are prescribed functions.

The stress-strain constitutive relations for isotropic viscoelastic material are given in differential form (Christensen 1982).

$$p_0 s_{ij} + p_1 \frac{ds_{ij}}{dt} + p_2 \frac{d^2 s_{ij}}{dt^2} + \cdots$$
$$= q_0 e_{ij} + q_1 \frac{de_{ij}}{dt} + q_2 \frac{d^2 e_{ij}}{dt^2} + \cdots \tag{18}$$

where s_{ij} and e_{ij} are deviatoric components of stress and strain, respectively, and p_0, p_1, p_2, and q_0, q_1, q_2 are material constants.

4. Dynamic Analysis of a One-Dimensional Viscoelastic Rod

Haitan and Yan (2003) solved the one-dimensional viscoelastic rod problem using an element-free Galerkin method. Their results agree well with other solutions. For one-dimensional problems, the governing equation becomes $\sigma_{x,x} = \rho \partial^2 u/\partial t^2$. Consider a viscoelastic rod with length $L = 0.9$, density $\rho = 100$, and Kelvin model with $p_0 = 1$, $q_0 = 1$, and $q_2 = 2$ (Haitan and Yan 2003). The governing equation in terms of displacement becomes

$$q_0 \frac{\partial^2 u}{\partial x^2} + q_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial^2 x}\right) = \rho \frac{\partial^2 u}{\partial x^2} \tag{19}$$

The boundary and initial conditions are

$$u(t,0) = 0 \tag{20}$$

$$\sigma_x = \frac{\partial u(t,L)}{\partial x} = 0 \tag{21}$$

$$u(0,x) = \tilde{u}^0(x) = 0 \tag{22}$$

$$\frac{\partial u(0,x)}{\partial t} = \tilde{\upsilon}^0(x) = 0.75 \frac{x}{L} \tag{23}$$

To solve the resulting differential equation using the generalized finite differences, the finite difference mesh is to be defined first. The region from 0 to L is subdivided into a number of equal subregions (*Ndiv*) of length Δx defining $N = Ndiv + 2$ mesh points (nodes), *Ndiv+1* regular points, and a dummy point, point *Ndiv+2*, (which does not belong to the viscoelastic bar). The term $\partial^2 u/\partial x^2$ is to be calculated at each $P(x_i)$ for $i = 1 \cdots N$. When it is substituted in Eq. (22) gives a system of ordinary differential equations in the form

$$M\ddot{u} + C\dot{u} + Ku = F \tag{24}$$

This system needs to be modified to impose the boundary conditions given. The condition at $x = 0$ $u(t,0) = 0$ is taken into account by equating all the elements of the first column and first row of each matrix to zero and taking $M_{1,1} = 1$ and $F_1 = 0$. The other boundary condition at $x = L$ $\left(\sigma_x = \partial u(t,L)/\partial x = 0\right)$ gives an extra equation at $x = L$ which can be used to eliminate the dummy unknown $u_N = u(t, L + \Delta x)$ from the system. The resulting second order system is converted into an equivalent system of first order ordinary differential equations,

$$\dot{z} = P(t,z) \tag{25}$$

where

$$z = \begin{pmatrix} u \\ \upsilon \end{pmatrix} \quad (26)$$

$$P = \begin{bmatrix} \dot{\upsilon} \\ M^{-1}\left(F - Ku - C\upsilon\right) \end{bmatrix} \quad (27)$$

which is then solved using a Runge-Kutta type solver.

5. Numerical Results

Free vibrations of the viscoelastic rod problem is solved using GFDM. The region is divided into *Ndiv* subregions yielding *N* regular nodes. The following two weight functions are tested:

In these equations $d = \sqrt{\left(x - x_i\right)^2}$ and *dm* is the radius of the circle (radius of influence) drawn at point $P\left(x_i\right)$ which determines the points in the neighborhood of *P* affecting i^{th} equation. It observed that both weight functions yield accurate results for $dm > 4\min\left(x_{i+1} - x_i\right)$. The results obtained are given together with the finite element (FE) and finite difference (FD) solutions using similar meshes. In the finite element solutions, linear elements are used. And simplest central differences are used in the finite difference solutions. It is observed that the precision of the generalized finite differences is similar to that of finite elements and finite differences. In Fig. 1, variation of tip displacement, obtained using GFDM, with time is given for different number of mesh points. A rapid convergence is observed. Results are presented for only number of mesh points equal to 3 and 24. In Fig. 2, a GFD solution for end displacement with time is given together with a FD solution. Finally, Fig. 3 shows the tip displacement solutions for N=24 using GFD, FD, and FE methods. The figure shows that all the results obtained are in good agreement.

Polynomial weight function (quadratic)

$$\omega_i = \begin{cases} 1 - 6\left(\dfrac{d}{dm}\right)^2 + 8\left(\dfrac{d}{dm}\right)^3 - 3\left(\dfrac{d}{dm}\right)^4 & for\ d \le dm \\ 0 & for\ d > dm \end{cases} \quad (28)$$

Polynomial weight function (cubic)

$$\omega_i = \begin{cases} \dfrac{2}{3} - 4\left(\dfrac{d}{dm}\right)^2 - 4\left(\dfrac{d}{dm}\right)^3 & for \quad d \le \dfrac{1}{2}dm \\ \dfrac{4}{3} - 4\left(\dfrac{d}{dm}\right) + 4\left(\dfrac{d}{dm}\right)^2 - \dfrac{4}{3}\left(\dfrac{d}{dm}\right)^3 & for \quad \dfrac{1}{2}dm \le d \le dm \\ 0 & for \quad d > dm \end{cases} \quad (29)$$

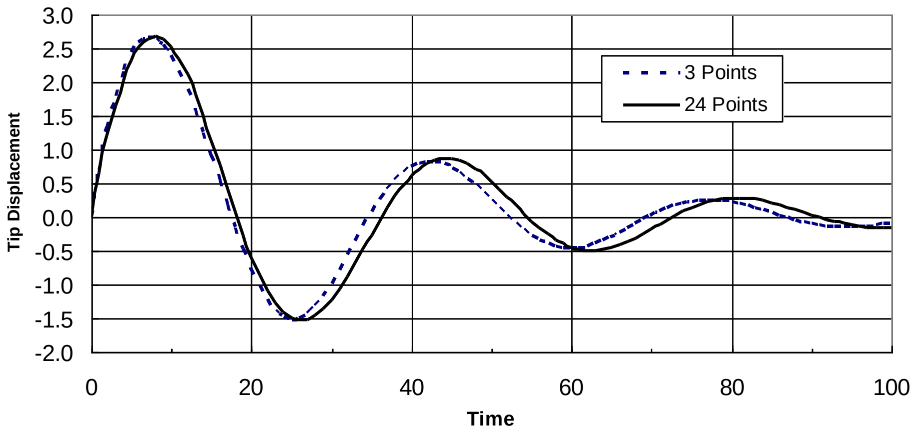

Fig. 1. The generalized finite difference (GFD) solutions for various number of nodes.

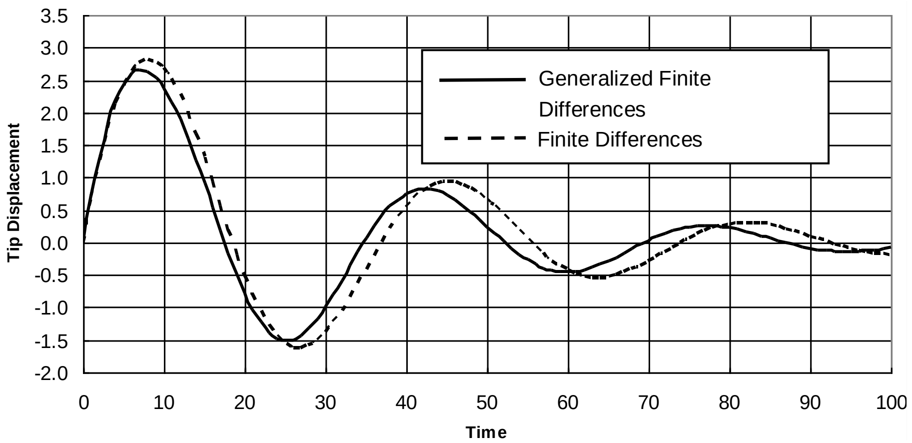

Fig. 2. The generalized finite difference (GFD) and the finite difference (FD) solutions for *N*=3.

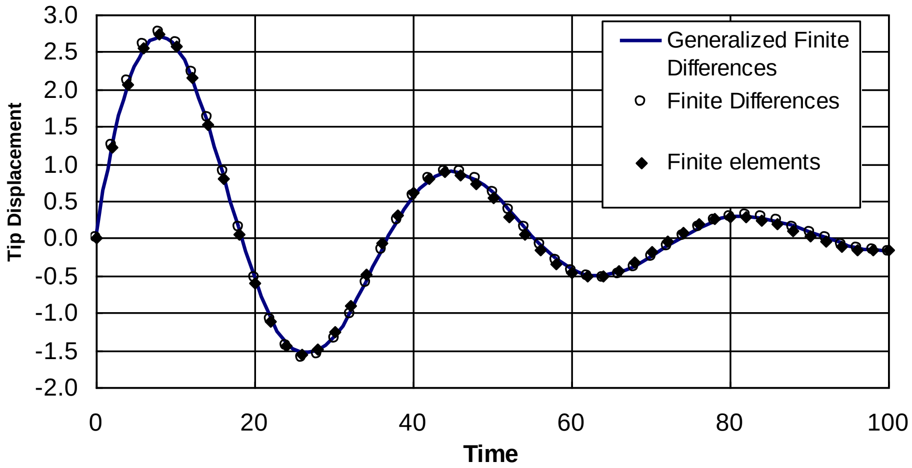

Fig. 3. Tip displacements for N=24.

6. Conclusions

GFDM has very obvious advantages over the finite element and classical finite difference methods when the domain of definition of the problem gets complicated. It is because of the fact that the GFDM does not require any element connectivity information; rather, it simply requires the coordinates of each point within the region and on its boundary together with the boundary condition information. Moreover, the type of boundary condition does not make the solution process any difficult. Both Dirichlet and Neumann-type boundary conditions can be imposed with ease. In this study, two different polynomial weight functions are tested, and it is observed that they result in solutions that are reasonably comparable. It is also observed that there is a fast improvement in the results with an increasing number of mesh points, provided that these points are evenly distributed within the region. The results obtained from GFDM, FD, and FE solutions were analyzed in this study. It is important to note that the results obtained from GFDM were observed to be strikingly similar to each other.

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Conflict of Interest

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